MOTION IN A PLANE (2D) — Chapter-3

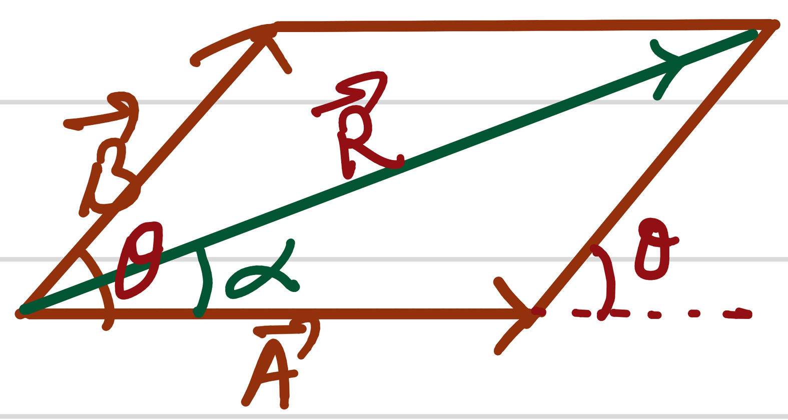
FORMULA SHEET

1. Unit vector:

$$\hat{A} = \frac{\vec{A}}{|\vec{A}|} = \frac{\vec{A}}{A} \qquad |\vec{A}| = A = \text{magnitude of } \vec{A}$$

2. Parallelogram law of vector addition

→ Magnitude of resultant vector of two vectors $\vec{A}$ and $\vec{B}$ inclined at an angle θ



$$R = \sqrt{A^2 + B^2 + 2AB\cos\theta}$$

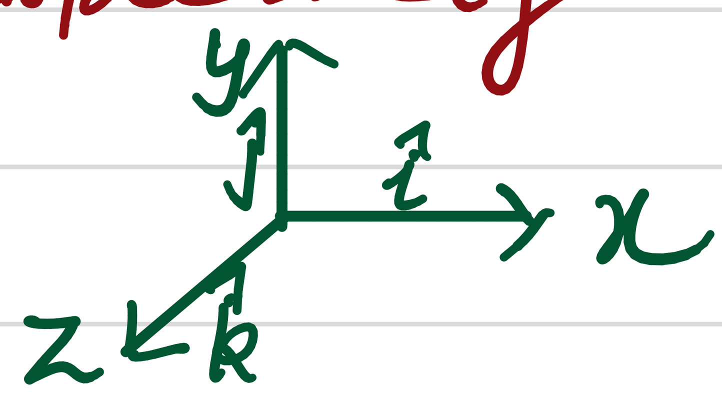
→ Direction of resultant vector with vector $\vec{A}$

$$\tan\alpha = \frac{B\sin\theta}{A + B\cos\theta}$$

3 Orthogonal triad of unit vectors: Base Vectors

→ $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors in the directions along x-axis, y-axis and z-axis respectively.

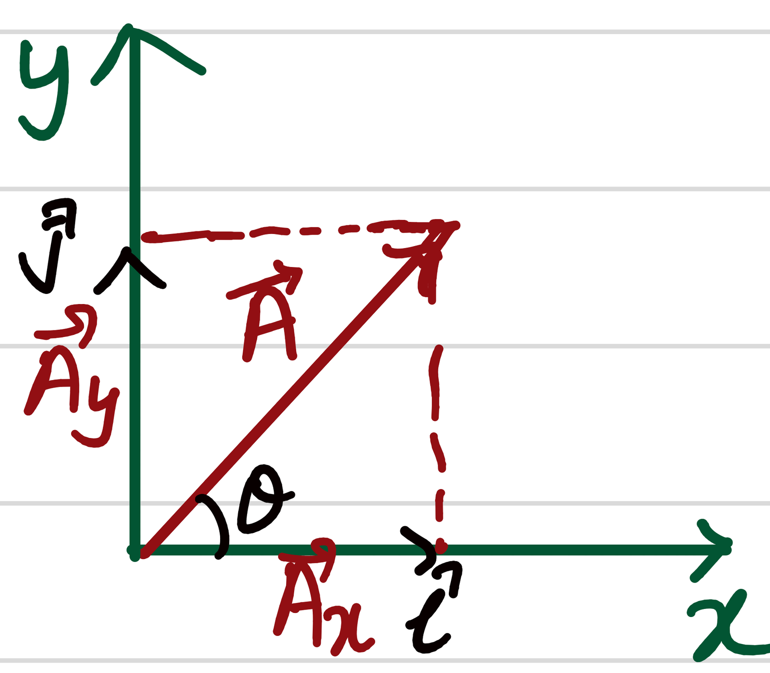
$$|\hat{i}| = |\hat{j}| = |\hat{k}| = 1$$



Rectangular components of a vector

$$\vec{A} = \vec{A}_x + \vec{A}_y$$

$$\vec{A} = A_x\hat{i} + A_y\hat{j}$$



also $A_x = A\cos\theta$, $A_y = A\sin\theta$

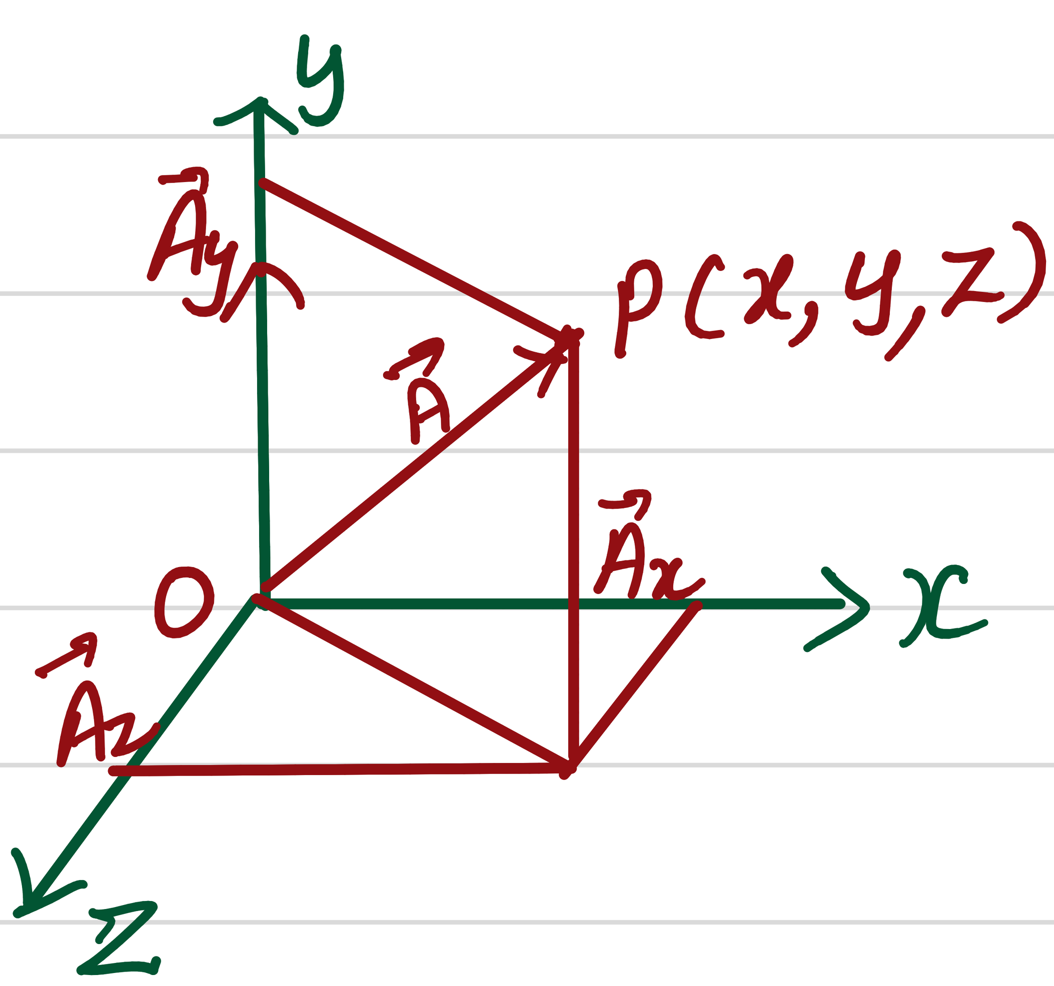
$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \tan\theta = \frac{A_y}{A_x}$$

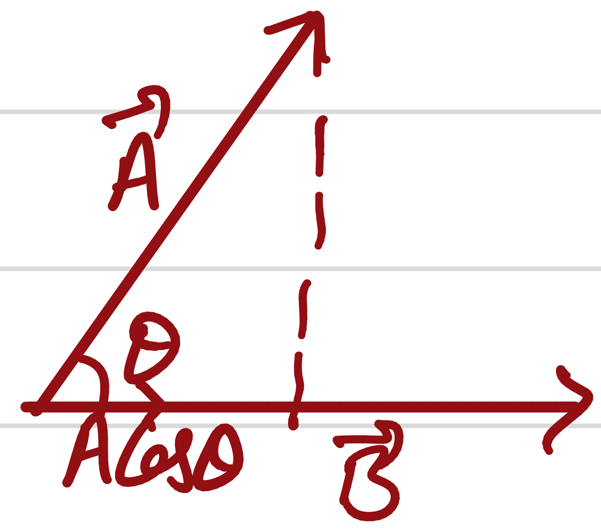
* For a vector in three dimensions

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$

Its magnitude

$$|\vec{A}| = A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$





Scalar or dot product

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta = AB\cos\theta$$

It can be +ve, -ve or zero depending upon θ.

* For parallel vectors
  $\theta = 0^\circ \quad \cos\theta = 1, \quad \vec{A}\cdot\vec{B} = AB$
* For antiparallel vectors
  $\theta = 180^\circ \quad \cos\theta = -1 \quad \vec{A}\cdot\vec{B} = -AB$
* For perpendicular vectors
  $\theta = 90^\circ \quad \cos\theta = 0 \quad \vec{A}\cdot\vec{B} = 0$
* $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$
* $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$
* $\vec{A}\cdot\vec{A} = A^2$
* $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$
* $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

→ In cartesian co-ordinates

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$$
$$\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$$

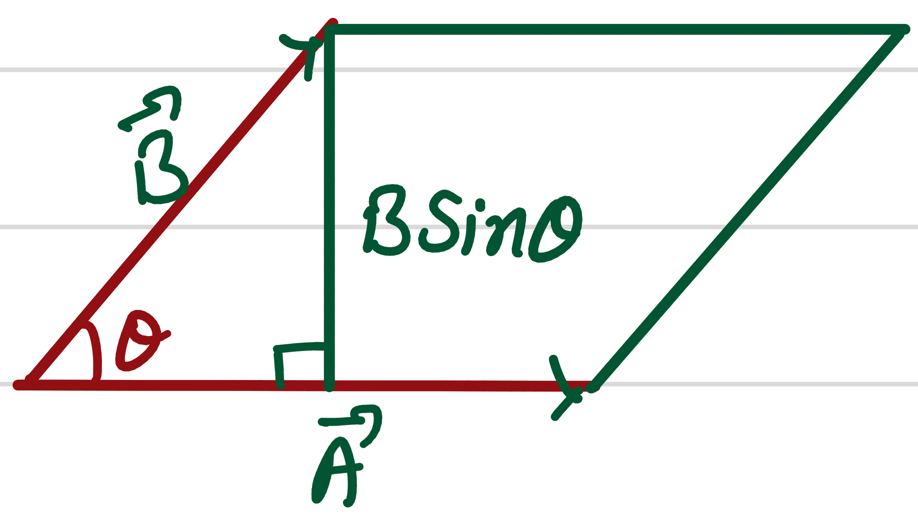
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{A_xB_x + A_yB_y + A_zB_z}{\sqrt{A_x^2+A_y^2+A_z^2}\sqrt{B_x^2+B_y^2+B_z^2}}$$

Vector product

$$\vec{A}\times\vec{B} = AB\sin\theta\,\hat{n}$$

$\hat{n}$ → unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$
Its direction is determined by right hand thumb rule.

* Geometrical interpretation
  $|\vec{A}\times\vec{B}|$ = Area of parallelogram
  $= AB\sin\theta$

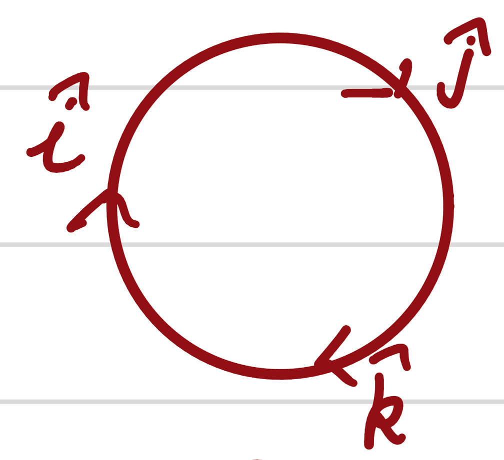


Properties of cross product

* For parallel or antiparallel vectors $\theta = 0^\circ$ or 180°

$$\vec{A} \times \vec{B} = \vec{0}$$

* $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
* $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
* $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0$
* $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{i} = \hat{j}$



* Unit vector perpendicular to the plane of $\vec{A}$ and $\vec{B}$ is given by

$$\hat{n} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|}$$

* Angle between $\vec{A}$ and $\vec{B}$

$$\sin\theta = \frac{\vec{A} \times \vec{B}}{|\vec{A}||\vec{B}|}$$

→ In cartesian coordinates

$$\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \hat{i}(A_y B_z - A_z B_y) - \hat{j}(A_x B_z - A_z B_x) + (A_x B_y - A_y B_x)\hat{k}$$

Position and displacement vector

Position vector

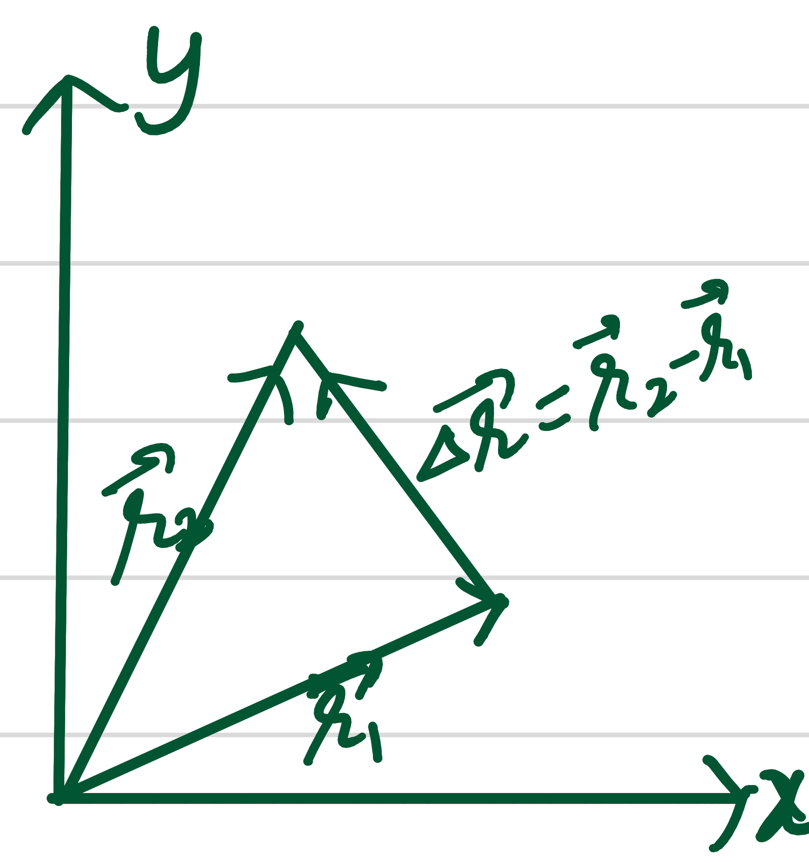
$$\vec{r} = x\hat{i} + y\hat{j}$$

Displacement vector

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$= (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$$\Delta\vec{r} = \Delta x \hat{i} + \Delta y \hat{j}$$



Velocity vector

$$\vec{v} = \frac{\Delta \vec{r}}{\Delta t}$$

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d\vec{r}}{dt}$$

In components form

$$v = v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$$

here $v_x = \frac{dx}{dt}$, $v_y = \frac{dy}{dt}$ $v_z = \frac{dz}{dt}$

Magnitude $|\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Acceleration vector

$$\vec{a} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta v}{\Delta t}$$

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

In component form

$$\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$$

where $a_x = \frac{dv_x}{dt}$, $a_y = \frac{dv_y}{dt}$, $a_z = \frac{dv_z}{dt}$

$$|\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

Equations of motion in two dimensions (a is constant)

	Along x-axis	Along y-axis
1.	$v_x = u_x + a_x t$	$v_y = u_y + a_y t$
2.	$x = u_x t + \frac{1}{2} a_x t^2$	$y = u_y t + \frac{1}{2} a_y t^2$
3.	$v_x^2 = u_x^2 + 2a_x x$	$v_y^2 = u_y^2 + 2a_y y$

* The motion in a plane with uniform acceleration can be treated as the superposition of two seperate simultaneous 1 D motions along two perpendicular dir's.

*Not in new syllabus

Relative velocity in two dimensions

(i) If two objects are moving with velocity $\vec{v}_A$ and $\vec{v}_B$ with respect to the ground, then

Relative velocity of A w.r.to B,

$$\vec{v}_{AB} = \vec{v}_A - \vec{v}_B$$

and relative velocity of B w.r.to A

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$

$$\vec{v}_{AB} = -\vec{v}_{BA} \Rightarrow |\vec{v}_{AB}| = |\vec{v}_{BA}|$$

$$\vec{v}_{AB} = \sqrt{v_A^2 + v_B^2 - 2v_A v_B \cos\theta}$$

Direction of $\vec{v}_{AB}$ with $\vec{v}_A$

$$\tan\beta = \frac{v_B \sin\theta}{v_A - v_B \cos\theta}$$

Projectile Motion:

air resistance is neglected

Horizontal Projectile

(i) Position of the projectile after time t:

$$x = ut \quad , \quad y = \frac{1}{2} g t^2$$

(ii) Equation of trajectory.

$$y = \frac{g}{2u^2} x^2$$

(iii) Velocity after time t

$$v = \sqrt{u^2 + g^2 t^2} \quad , \quad \beta = \tan^{-1}\left(\frac{gt}{u}\right)$$

(iv) Time of flight.

$$T = \sqrt{\frac{2h}{g}}$$

(V) Horizontal range

$$R = u \times T = u\sqrt{\frac{2h}{g}}$$

Projectile fired at an angle θ (Angular/oblique projectile)

(I) Components of initial velocity

$$u_x = u\cos\theta \qquad u_y = u\sin\theta$$

(II) Components of acceleration at any instant

$$a_x = 0 \quad , \quad a_y = -g$$

(III) Position after time t

$$x = (u\cos\theta)t \qquad y = (u\sin\theta)t - \frac{1}{2}gt^2$$

(IV) Equation of trajectory

$$y = x\tan\theta - \frac{g}{2u^2\cos^2\theta}x^2$$

(V) Maximum height,

$$H = \frac{u^2\sin^2\theta}{2g} \qquad H_{max} = \frac{u^2}{2g} \quad [\text{For } \theta = 90^\circ]$$

(VI) Time of flight,

$$T = \frac{2u\sin\theta}{g}$$

(VII) Horizontal Range,

$$R = \frac{u^2\sin^2 2\theta}{g}$$

(VIII) Maximum horizontal range

At $\theta = 45^\circ$

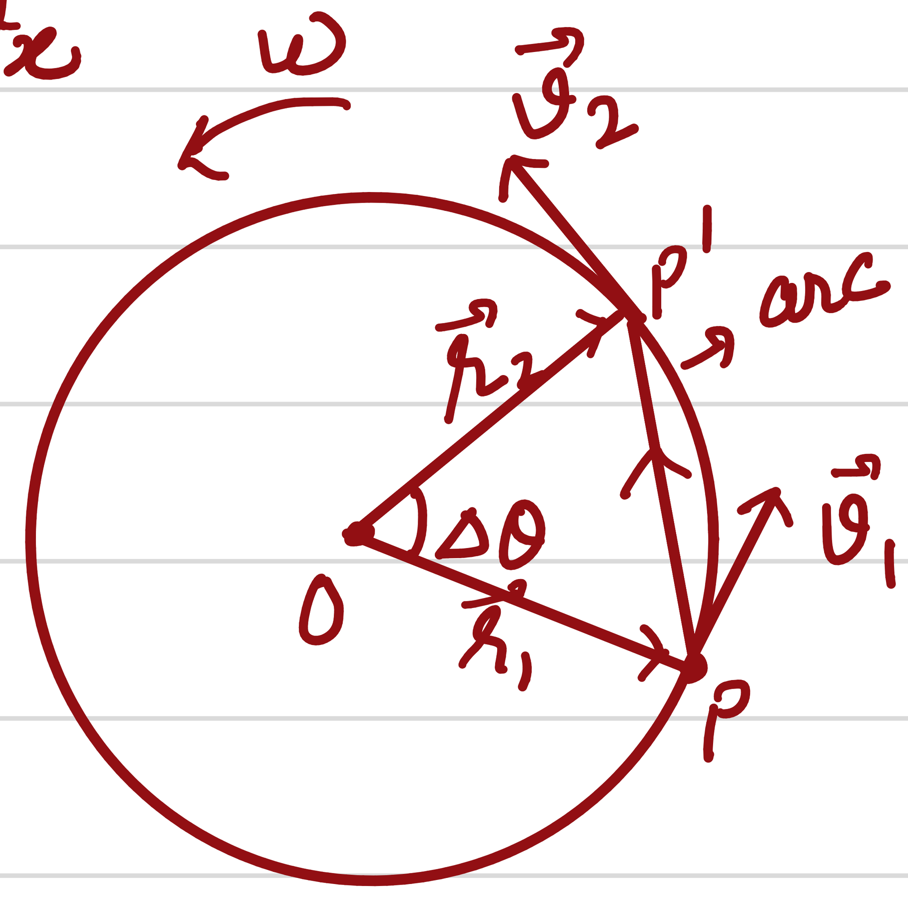
$$R_{max} = \frac{u^2}{g}$$

(IX) Velocity after time t

$$v_x = u\cos\theta \quad , \quad v_y = u\sin\theta - gt$$

$$v = \sqrt{v_x^2 + v_y^2} \quad \text{and} \quad \tan\beta = \frac{v_y}{v_x}$$

Uniform Circular Motion



(I) Angular displacement

$$\theta = \frac{\text{Arc}}{\text{Radius}} = \frac{s}{r}$$

(II) Angular velocity

$$\omega = \frac{\theta}{t} \quad \text{or} \quad \omega = \frac{\theta_2 - \theta_1}{t_2 - t_1}$$

(III) Time period and frequency

$$\omega = \frac{2\pi}{T} = 2\pi\nu$$

(IV) Relationship between v and ω

$$v = r\omega$$

Linear velocity = Radius x angular velocity

(V) Angular acceleration and its relation with linear acceleration

$$\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$$

$\alpha \rightarrow$ angular acceleration

Also, $a = r\alpha$

Linear acceleration = Radius x angular acceleration

(VI) Centripetal acceleration and centripetal force

$$a = \frac{v^2}{r} = r\omega^2, \quad F = \frac{mv^2}{r} = mr\omega^2$$

Important points

→ In horizontal projectile

- Initial vertical velocity $u_y = 0$
- Horizontal velocity is constant throughout the motion because no horizontal force acts.

- In vertical direction motion is – free fall.
- Time of flight depends only on height.
- Vertical velocity during the fall increases due to gravity.

→ In angular projectile

- Horizontal velocity is constant because no force is in horizontal direction.
- At highest point vertical velocity is zero.
- At highest point acceleration is not zero it is g always acts downward. ($g = 9.8\ m/s^2$)
- Time of ascent = Time of descent (In ideal condition) It is symmetric about the highest point.
- At highest point total velocity is minimum (only horizontal velocity remains)

→ Uniform circular motion

- In UCM speed is constant but velocity changes due to change in direction.
- UCM is an accelerated motion ⇒ centripetal acceleration
- In UCM acceleration is toward the centre of circle always (centripetal acceleration)
- Direction of velocity in UCM is tangential.
- In UCM an object cannot be in equilibrium because there is net acceleration towards center.
- Kinetic energy in UCM is constant because speed is constant.
- If centripetal force is removed suddenly, object will move tangentially to the circle in straight line (Newton's 1st law: Law of inertia)