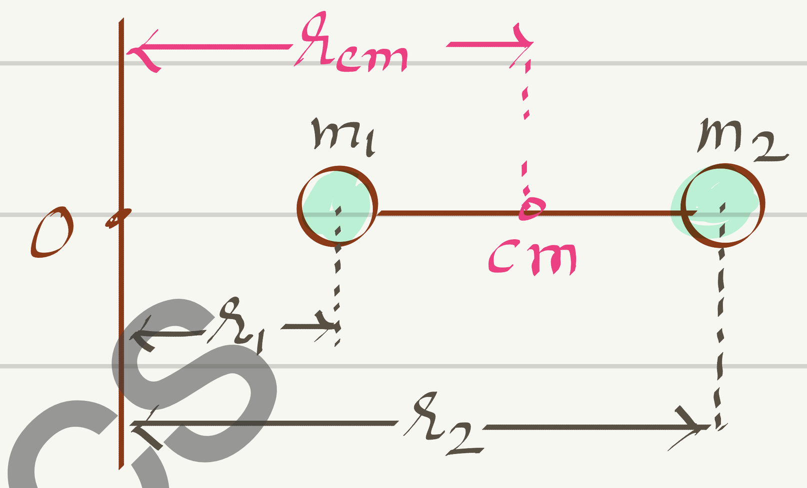
SYSTEM OF PARTICLE AND ROTATIONAL MOTION

Rigid body: A rigid body is a body with a perfectly and unchanging shape. The distance between the particles of a rigid body do not change under external force.
e.g. diamonds, steel beams, a ball bearing etc.

Centre of mass: The centre of mass of a body or system is defined as the point at which the total mass of the body is supposed to be concentrated.

Two particle system:

Position vector of centre of mass



$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

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$m_1, m_2 \rightarrow$ masses of particles
$\vec{r}_1, \vec{r}_2 \rightarrow$ position vector

* Centre of mass of two particle system lies closer to the heavier particle.
* If $m_1 = m_2$, then $\vec{r}_{cm} = \frac{\vec{r}_1 + \vec{r}_2}{2}$

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N Particle system:

$$\vec{r}_{cm} = \frac{m_1 r_1 + m_2 r_2 + \cdots m_n r_n}{m_1 + m_2 + \cdots m_n}$$

$$= \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i}$$

where $\sum_{i=1}^{n} m_i = M$ (mass of the system)

In component form:

$$\vec{r}_{cm} = x_{cm}\hat{i} + y_{cm}\hat{j} + z_{cm}\hat{k}$$

where $x_{cm} = \frac{\sum m_i x_i}{\sum m_i}$, $y_{cm} = \frac{\sum m_i y_i}{\sum m_i}$, $z_{cm} = \frac{\sum m_i z_i}{\sum m_i}$

or $$x_{cm} = \frac{m_1 x_1 + m_2 x_2 + \cdots m_n x_n}{m_1 + m_2 + \cdots m_n}$$

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$$y_{cm} = \frac{m_1 y_1 + m_2 y_2 + \cdots m_n y_n}{m_1 + m_2 + \cdots m_n}$$

$$z_{cm} = \frac{m_1 z_1 + m_2 z_2 + \cdots m_n z_n}{m_1 + m_2 + \cdots m_n}$$

Velocity of centre of mass:

$$\vec{v}_{cm} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2 + \cdots m_n\vec{v}_n}{m_1 + m_2 + \cdots m_n}$$

Acceleration of centre of mass:

$$\vec{a}_{cm} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2 + \cdots m_n\vec{a}_n}{m_1 + m_2 + \cdots m_n}$$

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* $\vec{F}_{ext} = M\vec{a}_{cm}$

* If $\vec{F}_{ext} = 0$, then $\vec{a}_{cm} = 0 \Rightarrow \vec{v}_{cm} = \text{constant}$.

i.e If no external force act on the body, then centre of mass will have constant momentum. Its velocity is always constant and acceleration is zero. i.e

$$M v_{cm} = \text{constant}$$

* If the system has continious distribution of mass, treating the mass element dm at position $\vec{r}$ as a point mass then by integration

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm$$

so that $x_{cm} = \frac{1}{M}\int x\,dm$, $y_{cm} = \frac{1}{M}\int y\,dm$ and $z_{cm} = \frac{1}{M}\int z\,dm$

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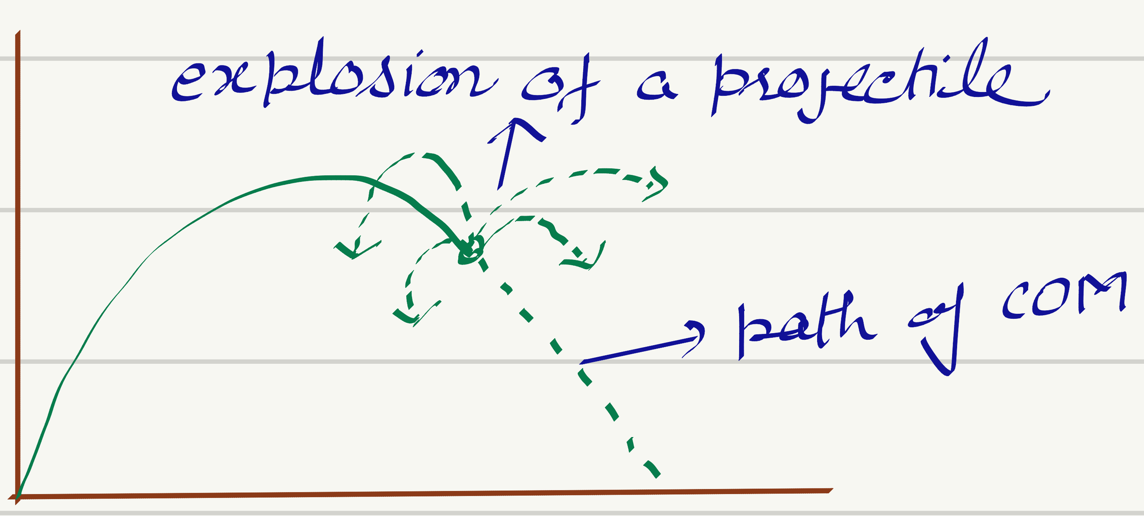
Examples of centre of mass motion:

(I) Earth and moon both move in circles about a common centre of mass. Since earth is hevier COM of the system of earth and moon is very close to earth. This COM revolves around the sun in an elliptical orbit.

(II) In radioactive decay the process caused by internal forces of the system. Therefore initial and final momentum are zero and hence decay products fly off in the opposite directions. The centre of mass remains at rest.

(III) Explosion of a projectile (e.g fire cracker) in mid air. When a projectile explodes in mid air, after explosion each fragment moves along its own parabolic path but centre of mass of the projectile continues to move in the same parabolic path.

Explanation - Explosion occur due to internal force, i.e $F_{ext} = 0$ Hence total momentum of the system is conserved. Thus path of



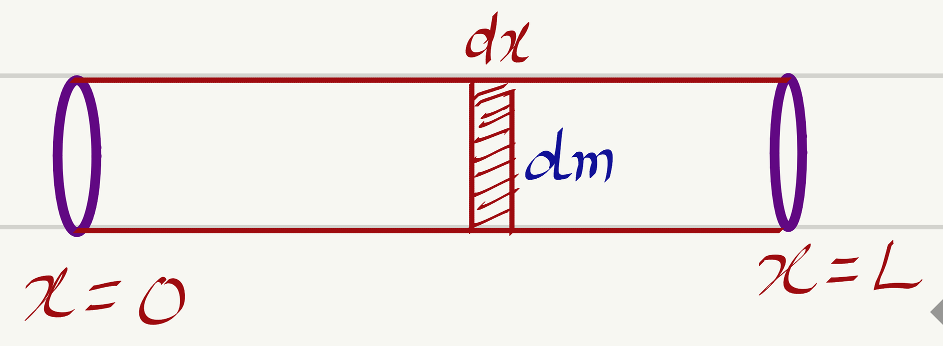
centre of mass remains unaffected and continues is parabolic path.

(iv) When a diver jump into water from a height, then body can move in any path but COM of his body moves in parabolic path. So centre of mass follows laws of motion.

Centre of mass of a uniform rod:

Suppose a rod of mass M and length L is lying along x axis with its one end at $x=0$ and the other at $x=L$.

Mass per unit length of the rod, $\lambda = \frac{M}{L}$ [λ = linear mass density]



Let dm, is the mass of element dx situated at distance x from the origin.

$$dm = \frac{M}{L}dx$$

Now, $$x_{cm} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^L x\frac{M}{L}dx}{M} \qquad \int dM = M$$

$$= \frac{1}{L}\int_0^L x\,dx$$

$$= \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$$= \frac{1}{2L}\left[L^2 - 0\right]$$

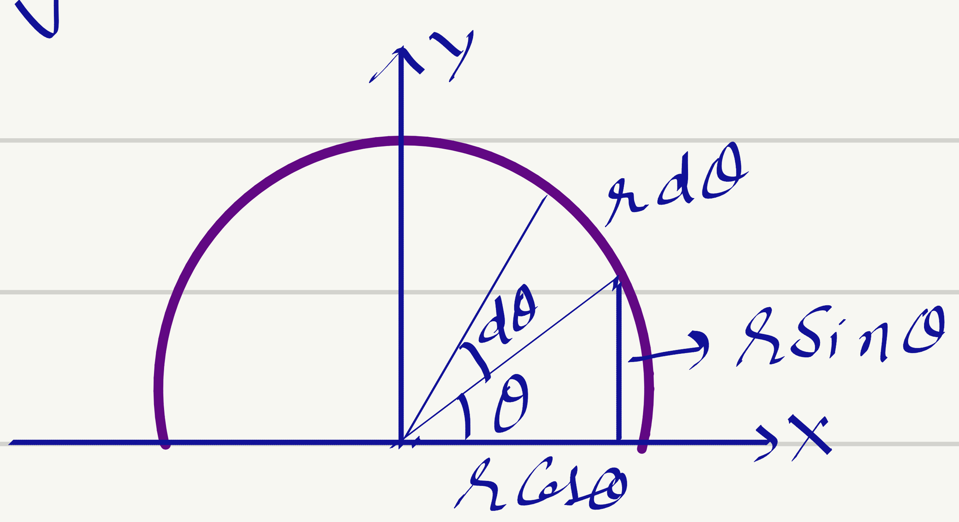
$$= \frac{L^2}{2L}$$

$$\boxed{x_{cm} = \frac{L}{2}} \qquad [y_{cm} = 0 \text{ and } z_{cm} = 0]$$

Position $\frac{L}{2}$ is the geometric centre of the rod.

The COM is at $(\frac{L}{2}, 0, 0)$

* Centre of mass of a semicircular wire is $\left(0, \frac{2R}{\pi}\right)$



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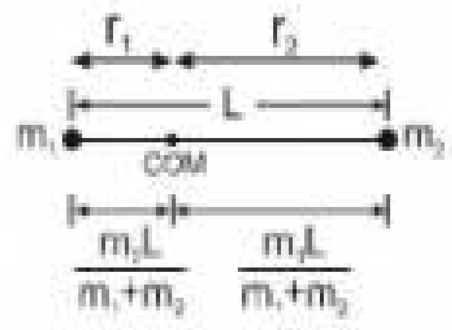
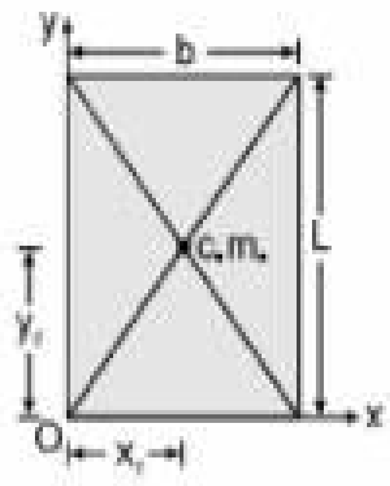
CENTRE OF MASS OF SOME COMMON SYSTEMS

A system of two point masses $m_1 r_1 = m_2 r_2$
The centre of mass lies closer to the heavier mass.



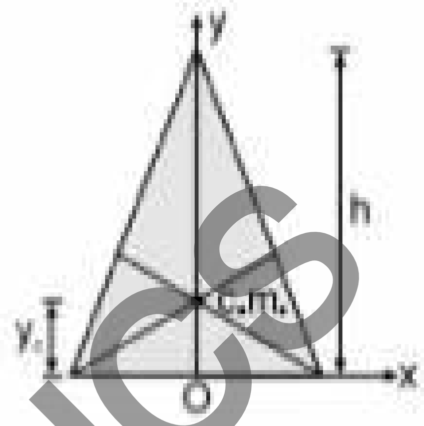
Rectangular plate (By symmetry)

$x_c = \frac{b}{2}$ $\quad y_c = \frac{L}{2}$



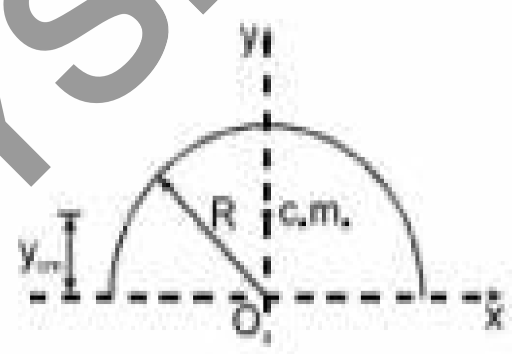
A triangular plate (By qualitative argument)

at the centroid : $y_c = \frac{h}{3}$



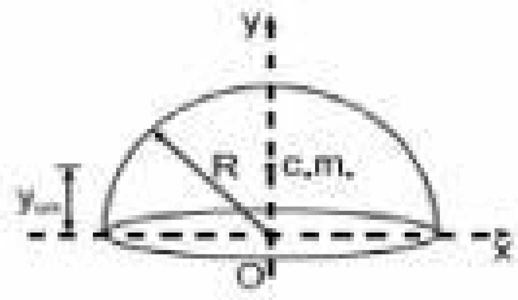
A semi-circular ring

$y_c = \frac{2R}{\pi}$ $\quad x_c = 0$



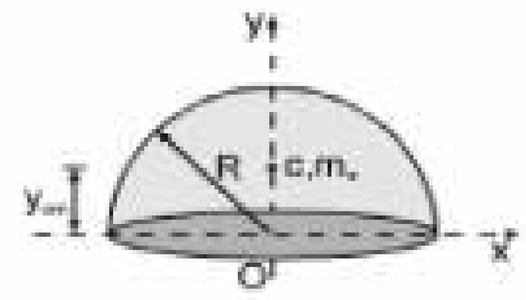
A hemispherical shell

$y_c = \frac{R}{2}$ $\quad x_c = 0$



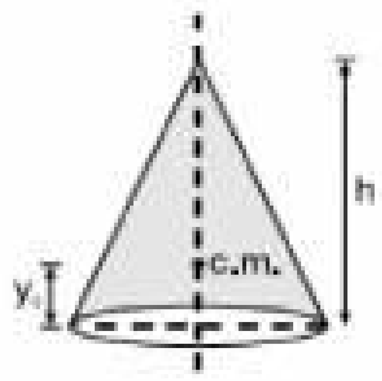
A solid hemisphere

$y_c = \frac{3R}{8}$ $\quad x_c = 0$



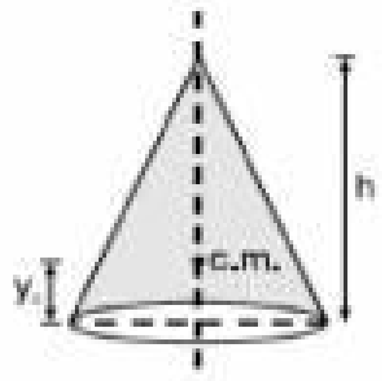
A circular cone (solid)

$y_c = \frac{h}{4}$



A circular cone (hollow)

$y_c = \frac{h}{3}$



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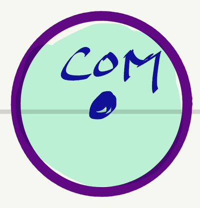
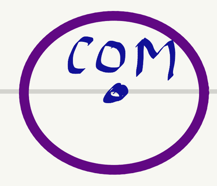
* The centre of mass may lie inside or outside the body. e.g. COM of a ring lies outside its body.



sphere
(COM is inside the body)



ring
(COM is in empty space, outsid the body)

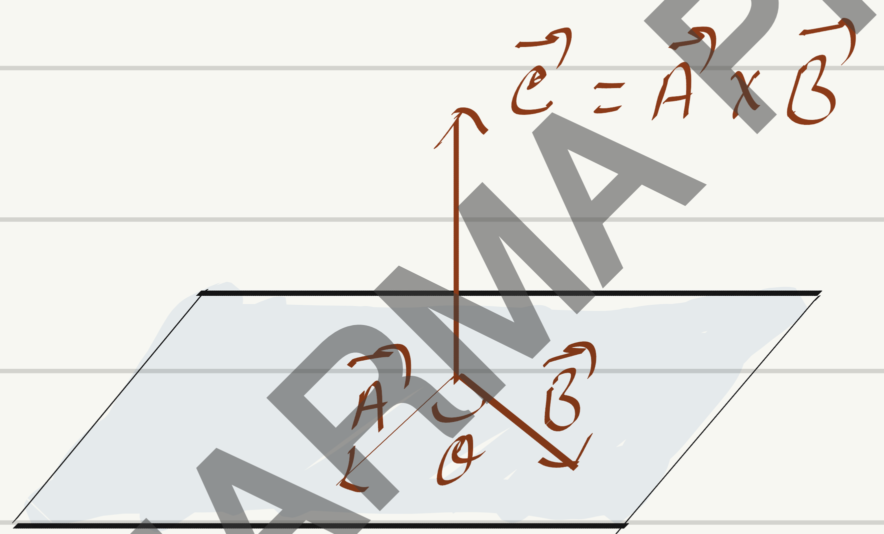
Vector product of two vectors

The vector or cross product of two vectors $\vec{A}$ and $\vec{B}$ represented as

$$\vec{A} \times \vec{B} = \vec{C} = AB\sin\theta\,\hat{n} \quad [\theta \text{ is angle b/w } \vec{A} \,\&\, \vec{B}]$$

where $\hat{n}$ is the unit vector in the dir^n of $\vec{C}$.

* $\vec{C}$ is perpendicular to the plane of $\vec{A} \times \vec{B}$.
* The dir^n of $\vec{C}$ is given by Right hand thumb rule.



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Properties of Cross Product:

(i) cross product is not commutative.

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

(ii) Distributive w.r. to vector addition.

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$

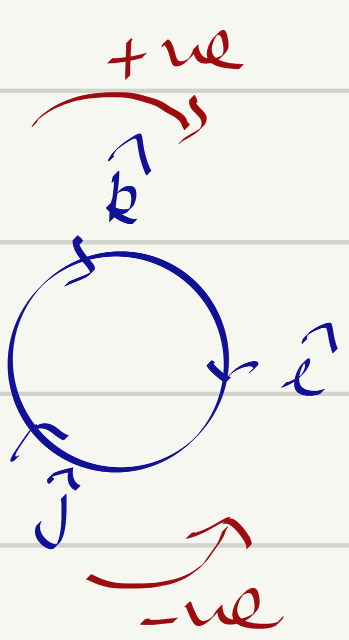
(iii) $\vec{A} \times \vec{A} = 0$ as $\vec{A} \times \vec{A} = AA\sin 0^\circ$

(iv) $\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \cdot \hat{k} = 0$ $[\sin 0^\circ = 0]$

and $\hat{i} \times \hat{j} = \hat{k}$, $\hat{j} \times \hat{i} = -\hat{k}$

$\hat{j} \times \hat{k} = \hat{i}$, $\hat{k} \times \hat{j} = -\hat{i}$

$\hat{k} \times \hat{i} = \hat{j}$, $\hat{i} \times \hat{k} = -\hat{j}$



* If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

and $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$

then
$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{i}(B_z A_y - B_y A_z) - \hat{j}(B_z A_x - B_x A_z) + \hat{k}(B_y A_x - B_x A_y)$$

* If $\vec{A}$ and $\vec{B}$ are perpendicular $[\theta = 90°]$

$$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}| \qquad [\sin 90 = 1]$$

* If $\vec{A}$ and $\vec{B}$ are parallel $[\theta = 0°]$

$$\vec{A} \times \vec{B} = 0 \qquad [\because \sin 0° = 0]$$

Q. Find the scalar and vector product of two vectors $\vec{a} = (3\hat{i} - 4\hat{j} + 5\hat{k})$ and $\vec{b} = (-2\hat{i} + \hat{j} - 3\hat{k})$

Ans.

Scalar product

$$\vec{a} \cdot \vec{b} = (3\hat{i} - 4\hat{j} + 5\hat{k}) \cdot (-2\hat{i} + \hat{j} - 3\hat{k})$$

$$= -6 - 4 - 15$$

$$\vec{a} \cdot \vec{b} = -25 \text{ unit}$$

Vector product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 5 \\ -2 & 1 & -3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \hat{i}(12 - 5) - \hat{j}(-9 + 10) + \hat{k}(3 - 8)$$

$$\vec{a} \times \vec{b} = (7\hat{i} - \hat{j} - 5\hat{k}) \quad \text{Ans}$$

Note:→ $\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})$

$$= -(7\hat{i} - \hat{j} - 5\hat{k})$$

$$= (-7\hat{i} + \hat{j} + 5\hat{k})$$

* Unit vector in the dirⁿ of a given vector $\vec{A}$ →

$$\hat{n} = \frac{\vec{A}}{|\vec{A}|}$$

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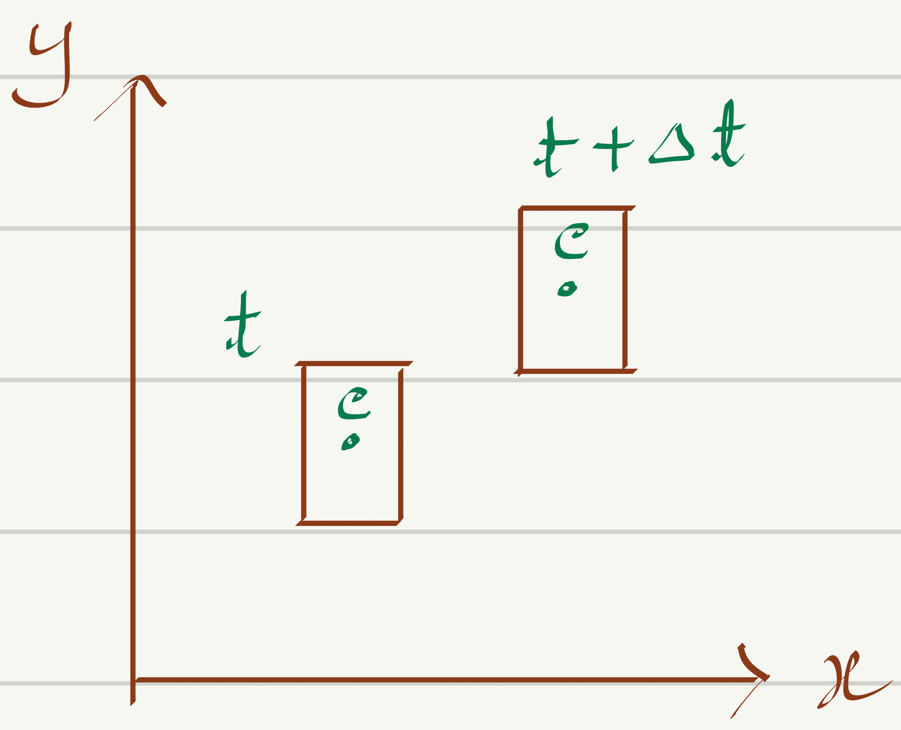
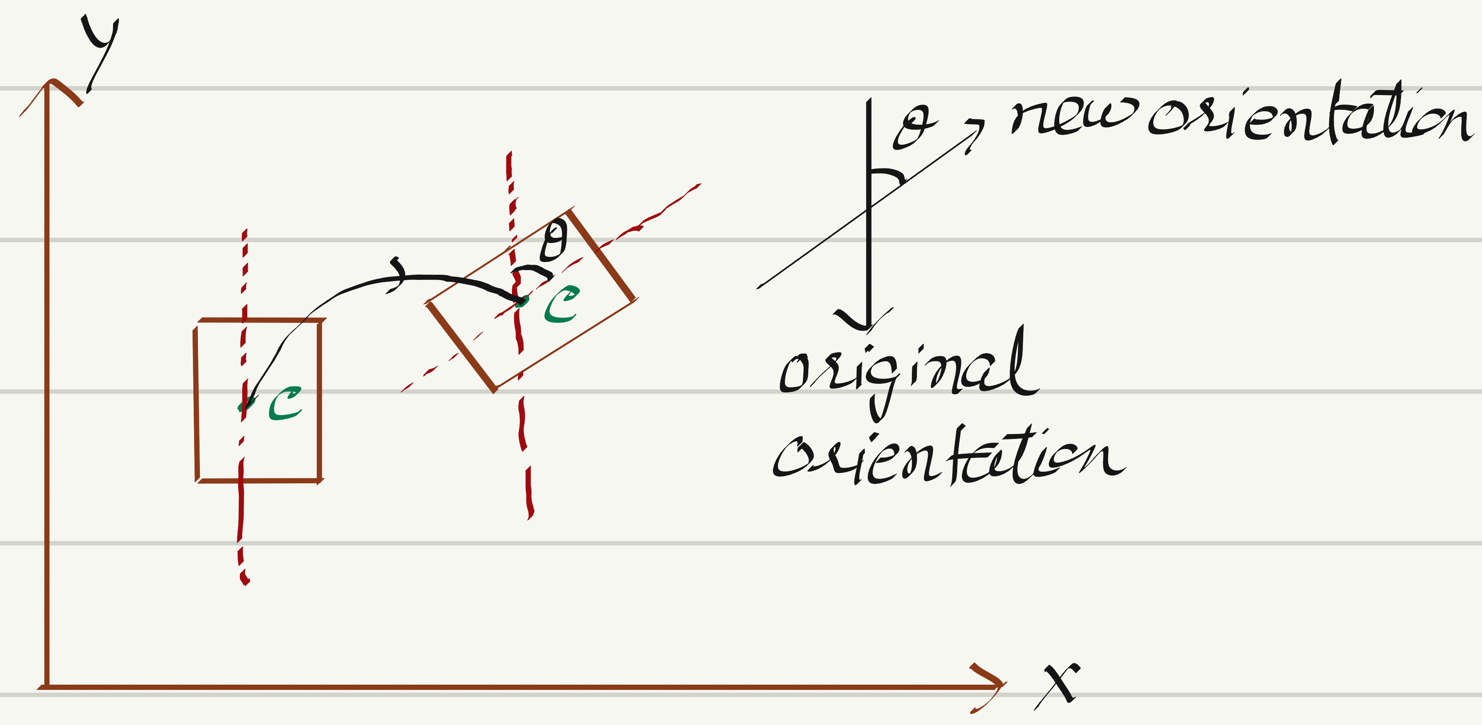
Rotational motion of a rigid body:

If a body changes its orientation during its motion it is said to be in rotational motion



Pure translation

(Point 'c' com does not change its orientation)



Combination of translation and rotation

(Point 'c' changes it orientation by θ)

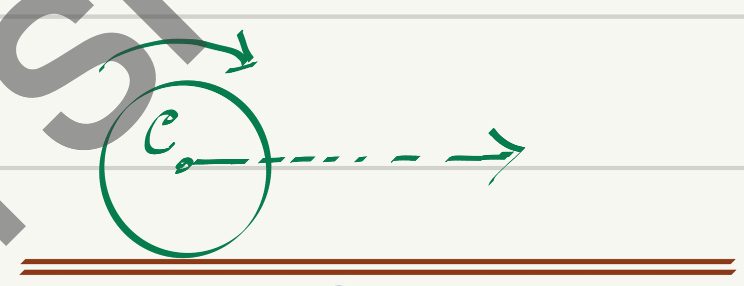
Types of motions involving rotation:

(I) Rotation about a fixed axis.

e.g. Rotation of ceiling fan, opening and closing doors and rotation of needles of wall clock etc.

(II) Rotation about an axis in translation:

e.g. Rolling of a wheel, moving on straight levelled road. fig (a)



(a)

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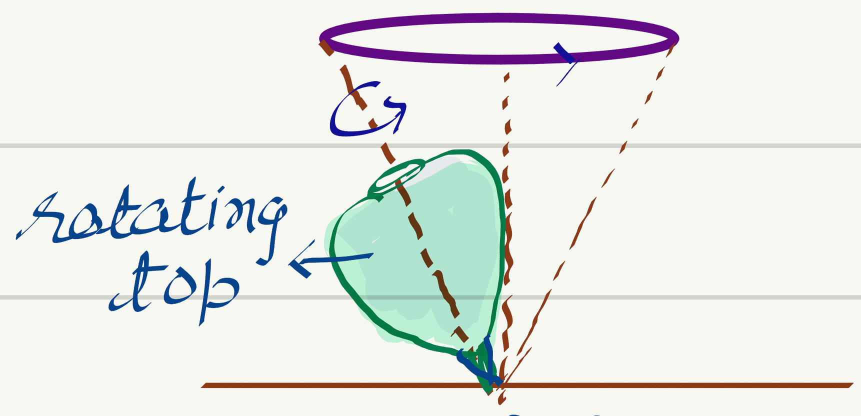


(b)

(III) Rotation about an axis in rotation:

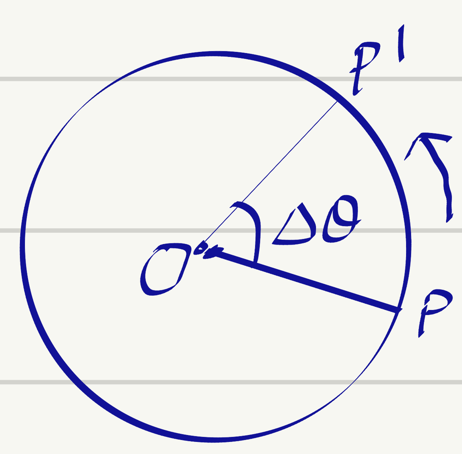
e.g. The top of rotating top, rotates about its central axis of symmetry and this axis sweeps a cone about a verticle axis. fig (b)

Kinematics of rotational motion:

Angular Displacement (θ) Angular displacement is the angle a particle rotates through.

Unit - radian, dimensionless

$$\text{Angle} = \frac{\text{Arc}}{\text{radius}}$$



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Angular velocity (ω)

The angular displacement per unit time is defined as angular velocity.

$$\omega = \frac{\Delta\theta}{\Delta t}$$

$\Delta\theta \rightarrow$ angular displacement in Δt time.

Instantaneous anglular velocity

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Average angular velocity

$$\omega_{av} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

unit $\rightarrow$ radian

Dimensions - $[M^0 L^0 T^{-1}]$ or $[T^{-1}]$ same as frequency

It is vector quantity. Its dir^n is given by right hand thumb rule.

$$\vec{v} = \vec{\omega} \times \vec{r}$$

and $\omega = \frac{2\pi}{T} = 2\pi\nu$

$T \rightarrow$ Time period

$\nu \rightarrow$ frequency

* Angular velocity can be +ve or −ve depending on increasing θ (anticlockwise) or decreasing θ (clockwise)
* Magnitude of angular velocity is called angular speed.

Angular Acceleration (α)

The rate of change of angular velocity is defined as angular acceleration.

$$\alpha = \frac{d\omega}{dt}$$

Unit - rad/s^2 Dim^n - $[M^0 L^0 T^{-2}]$ or $[T^{-2}]$

It is vector quantity. Dir^n - along the change in dir^n of ω.

* Average angular acc^n $\alpha = \frac{\omega_2 - \omega_1}{t_2 - t_1}$

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* Relation b/w angular accⁿ α and tangential accⁿ a_t

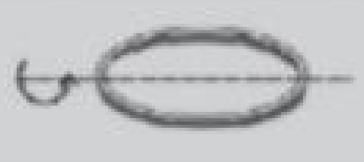
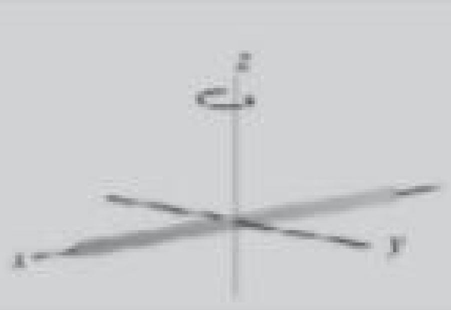
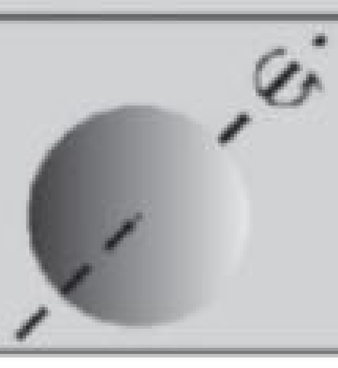
$$\vec{a}_t = \vec{\alpha} \times \vec{r}$$

* Net accⁿ $\vec{a} = \vec{a}_t + \vec{a}_r$

$$= \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

S. No.	Translational Motion	Rotational motion about a fixed axis
1.	Displacement x	Angular displacement θ
2.	Velocity $v = \frac{dx}{dt}$	Angular velocity $\omega = \frac{d\theta}{dt}$
3.	Acceleration $a = \frac{dv}{dt}$	Angular acceleration $\alpha = \frac{d\omega}{dt}$
4.	Mass M	Moment of inertia I
5.	Force $F = Ma$	Torque $\tau = I\alpha$
6.	Work $dW = Fds$	Work $dw = \tau d\theta$
7.	Kinetic energy $K = \frac{Mv^2}{2}$	Kinetic energy $K = \frac{I\omega^2}{2}$
8.	Power $P = Fv$	Power $P = \tau\omega$
9.	Linear momentum $P = Mv$	Angular momentum $L = I\omega$
10.	Equations of translatory motion $v = u + at$ $s = ut + \frac{1}{2}at^2$ $v^2 - u^2 = 2as$ where the symbols have their usual meaning	Equations of rotational motion $\omega = \omega_0 + \alpha t$ $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$ $\omega^2 - \omega_0^2 = 2\alpha\theta$ where the symbols have their usual meaning.
11.	Linear momentum is conserved if no external force acts on the system.	Angular momentum is conserved if no external torque acts on the system.

Table 7.1 Moments of Inertia of some regular shaped bodies about specific axes

Z	Body	Axis	Figure	I
(1)	Thin circular ring, radius R	Perpendicular to plane, at centre		MR^2
(2)	Thin circular ring, radius R	Diameter		$MR^2/2$
(3)	Thin rod, length L	Perpendicular to rod, at mid point		$ML^2/12$
(4)	Circular disc, radius R	Perpendicular to disc at centre		$MR^2/2$
(5)	Circular disc, radius R	Diameter		$MR^2/4$
(6)	Hollow cylinder, radius R	Axis of cylinder		MR^2
(7)	Solid cylinder, radius R	Axis of cylinder		$MR^2/2$
(8)	Solid sphere, radius R	Diameter		$2MR^2/5$

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Moment of Inertia (I):

The moment of inertia of a particle about an axis of rotation is equal to the product of its mass and square of its distance from the rotation axis.

$$I = mr^2$$

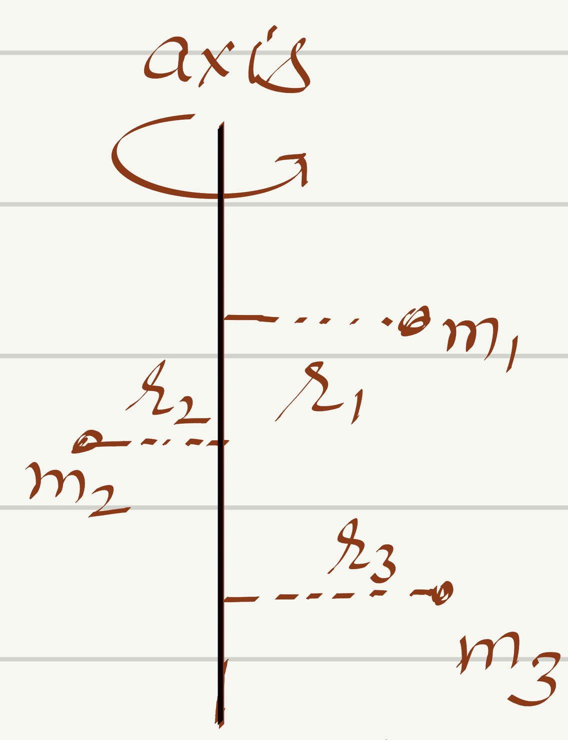
where r is the perpendicular distance from the axis of rotation

It is the measure of the property by virtue of which a body opposes any change in its rotational motion.

Moment of inertia of a system of particles

$$I = m_1 r_1^2 + m_2 r_2^2 + \cdots m_n r_n^2$$

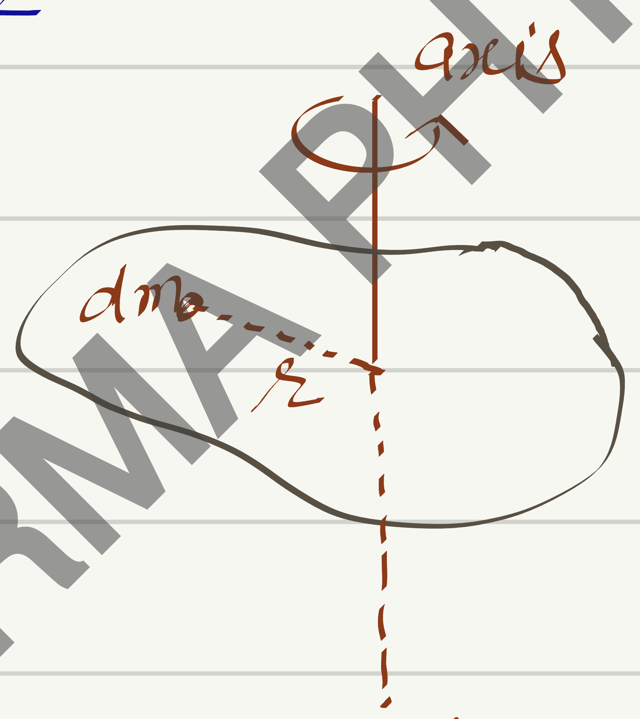
$$= \Sigma m r^2$$



deserete body

* For continous body,

$$I = \int r^2 dm$$



continuous body

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* Moment of inertia depends on -

(a) mass of the body

(b) mass distribution of the body

(c) position of axis of rotation

It does not depend on -

(a) angular velocity (b) angular accn (c) torque

(d) angular momentum

Unit → SI unit - $kg\,m^2$ CGS unit - $g\text{-}cm^2$

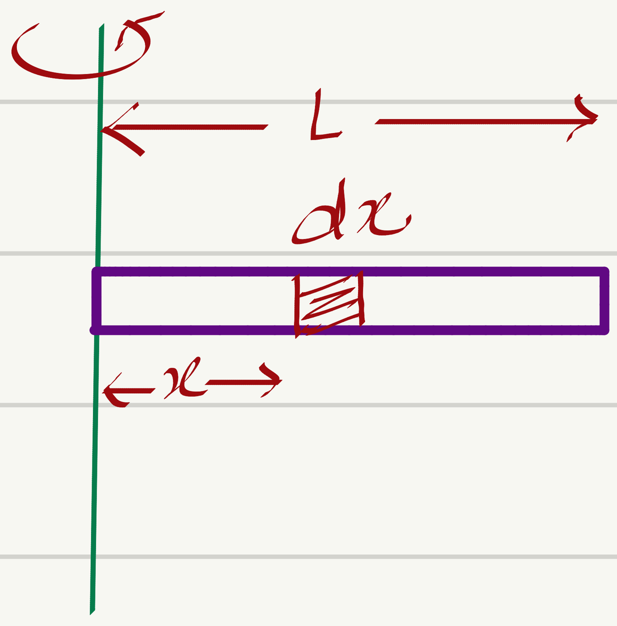
Dimn - $[M^1L^2T^0]$ or $[ML^2]$

It is scalar quantity

* As the distance mass increase from the rotation axis, the moment of inertia (M.I) increase.

Moment of Inertia of symmetrical mass distribution

(1) Moment of inertia of a rod about an axis passing through its end and perpendicular to length -



If the mass of the rod is M and mass of element is dm, then

$$dm = \frac{M}{L}dx$$

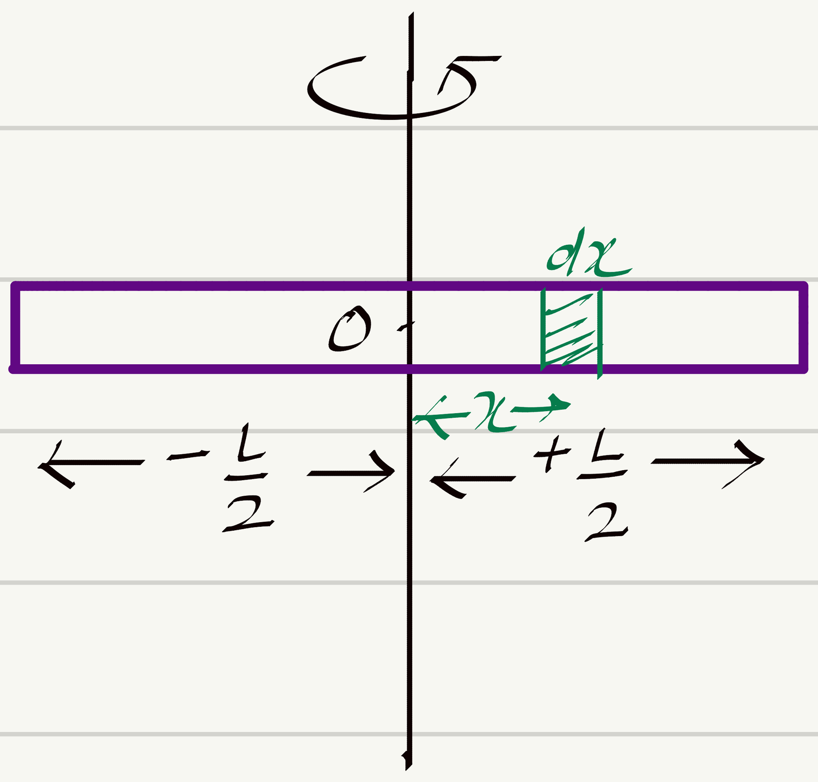
$$I = \int r^2 dm = \int_0^L x^2 \frac{M}{L} dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_0^L \qquad \left[\because \int x^2 dx = \frac{x^3}{3}\right]$$

or

$$\boxed{I = \frac{ML^3}{3}}$$

Moment of inertia of a rod about an axis passing through its centre of mass and perpendicular its length -



$$dm = \frac{M}{L}dx$$

$$I = \int r^2 dm$$

$$= \int_{-\frac{L}{2}}^{+\frac{L}{2}} x^2 \frac{M}{L} dx$$

$$= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-\frac{L}{2}}^{+\frac{L}{2}}$$

$$= \frac{M}{3L}\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right]$$

$$= \frac{M}{3L}\left[\frac{L^3}{8} + \frac{L^3}{8}\right]$$

$$= \frac{M \times \cancel{2}L^3}{3L \times \cancel{8}_4}$$

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$$\boxed{I = \frac{ML^2}{12} = \frac{1}{12} ML^2}$$

Radius of Gyration (K):

It is the distance from the rotation axis where the mass of the object could be assumed to be concentrated without altering the moment of inertia of the body about that axis.

If the mass m of the body were actually concentrated at a distance K from the axis, then moment of inertia about that axis,

$$I = mK^2$$

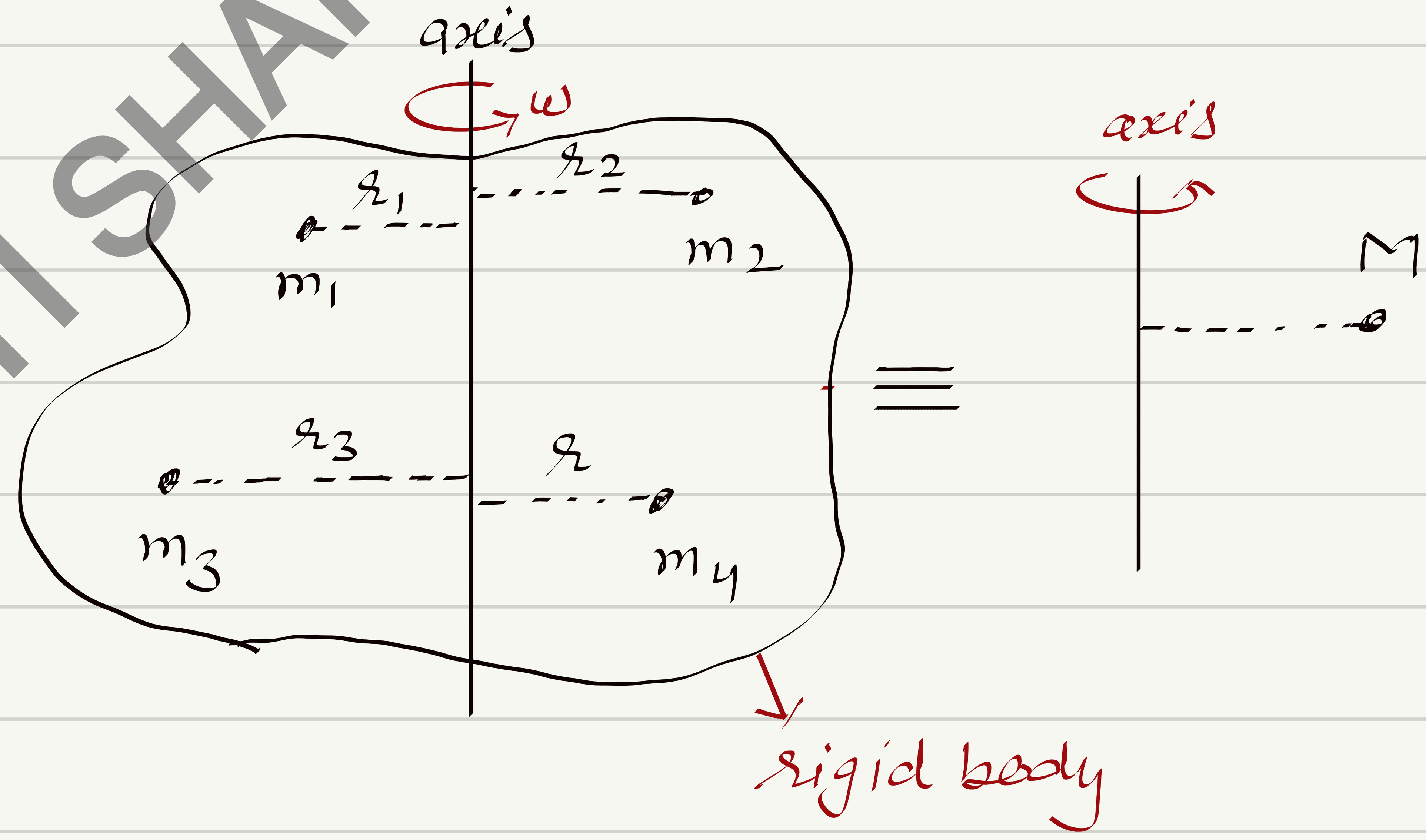
or $$\boxed{K = \sqrt{\frac{I}{m}}}$$

Unit - m

It is scalar quantity.

K has no meaning without axis of rotation

Ex.



here $$K = \sqrt{\frac{m_1 r_1^2 + m_2 r_2^2 + \cdots m_n r_n^2}{m_1 + m_2 + \cdots m_n}}$$

If $m_1 = m_2 \cdots = m$ then $M = mn$, $n \rightarrow$ total no. of particles

$$\boxed{K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots r_n^2}{n}}}$$

$[m_1 + m_2 + \cdots m_n = mn]$

* Radius of gyration depends on: (i) axis of rotation
  (ii) distribution of mass

It is independent of mass of the body.

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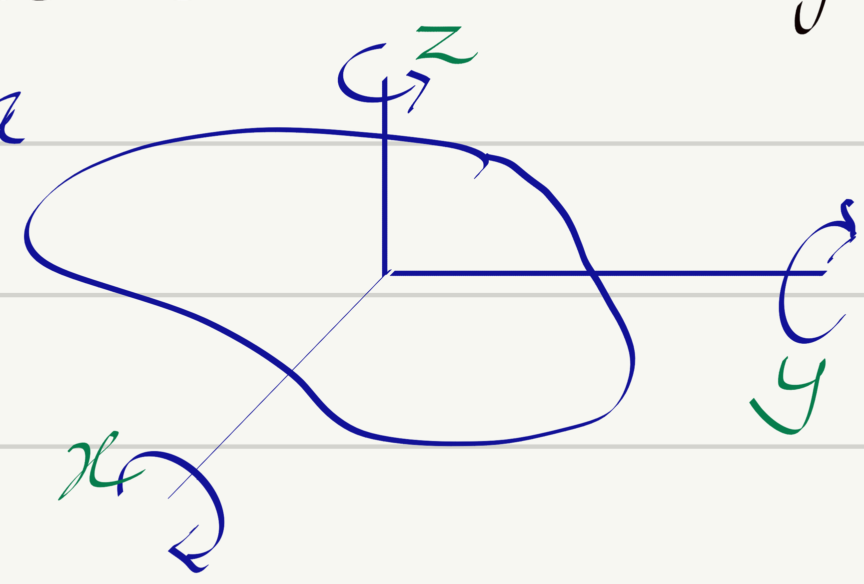
Theorem of moment of inertia

(1) Theorem of perpendicular axes [For plane lamina only]

The moment of inertia of a plane lamina about an axis perpendicular to its plane is equal to the sum of its moments of inertia about any two mutually perpendicular axes which intersects with the first axis.



$$\boxed{I_z = I_x + I_y}$$

It is applicable only for 2D bodies

where I_x = MI about x-axis
I_y = MI about Y-axis
I_z = MI about z-axis

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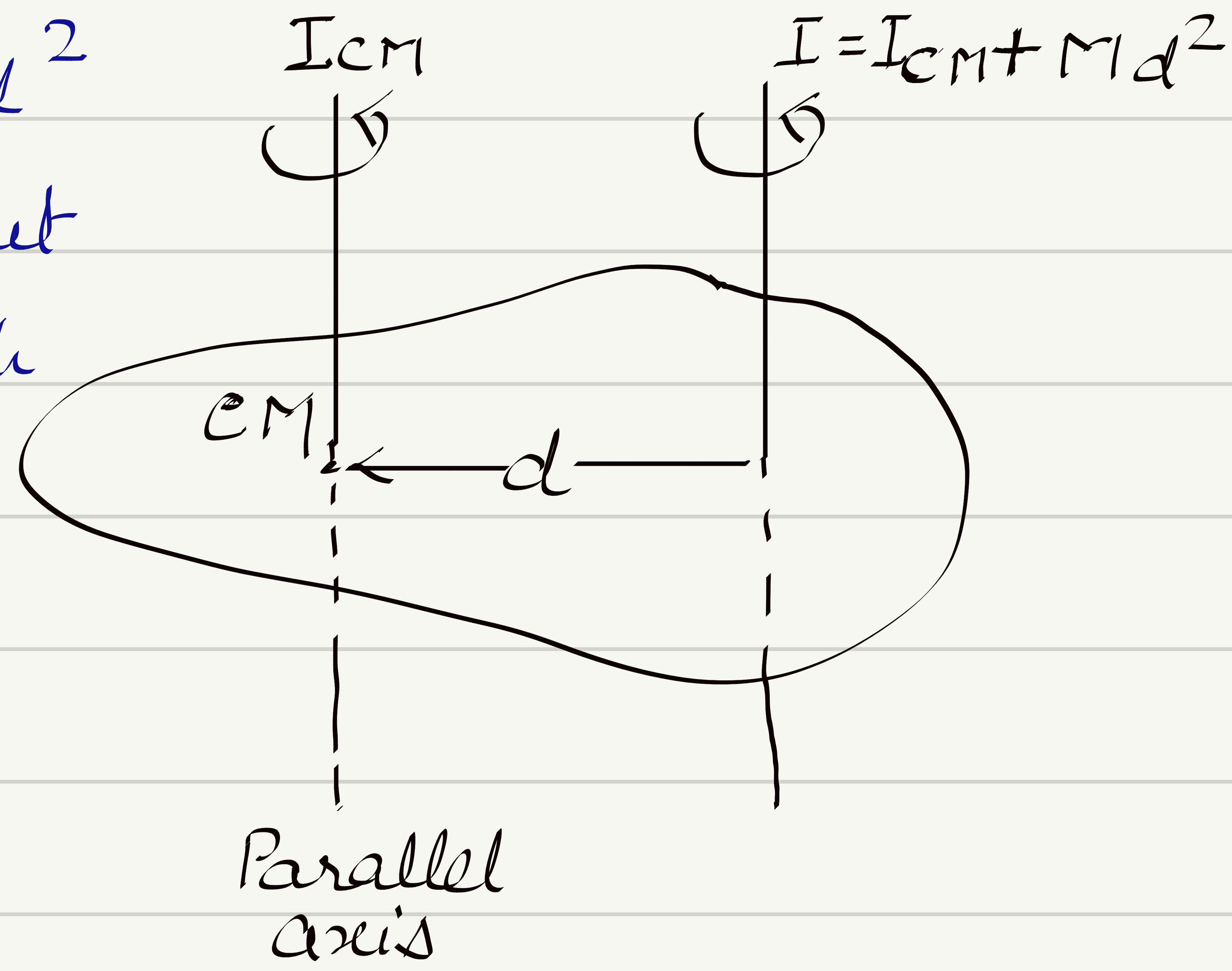
Theorem of parallel axes (For all type of bodies)

Moment of inertia of a body about any axis is equal to the moment of inertia about a parallel axis passing through the centre of mass plus product of mass of the body and the square of the distance between these two parallel axes.

$$I = I_{cm} + Md^2$$

here I_{cm} = moment of inertia about the axis passing through the centre of mass



Torque (τ): It is the rotational analogue of force.
Torque is defined as the vector product of the position vector and the force vector.
Mathematically, It is product of the force applied and the distance from the point of application of the force to the axis of rotation.

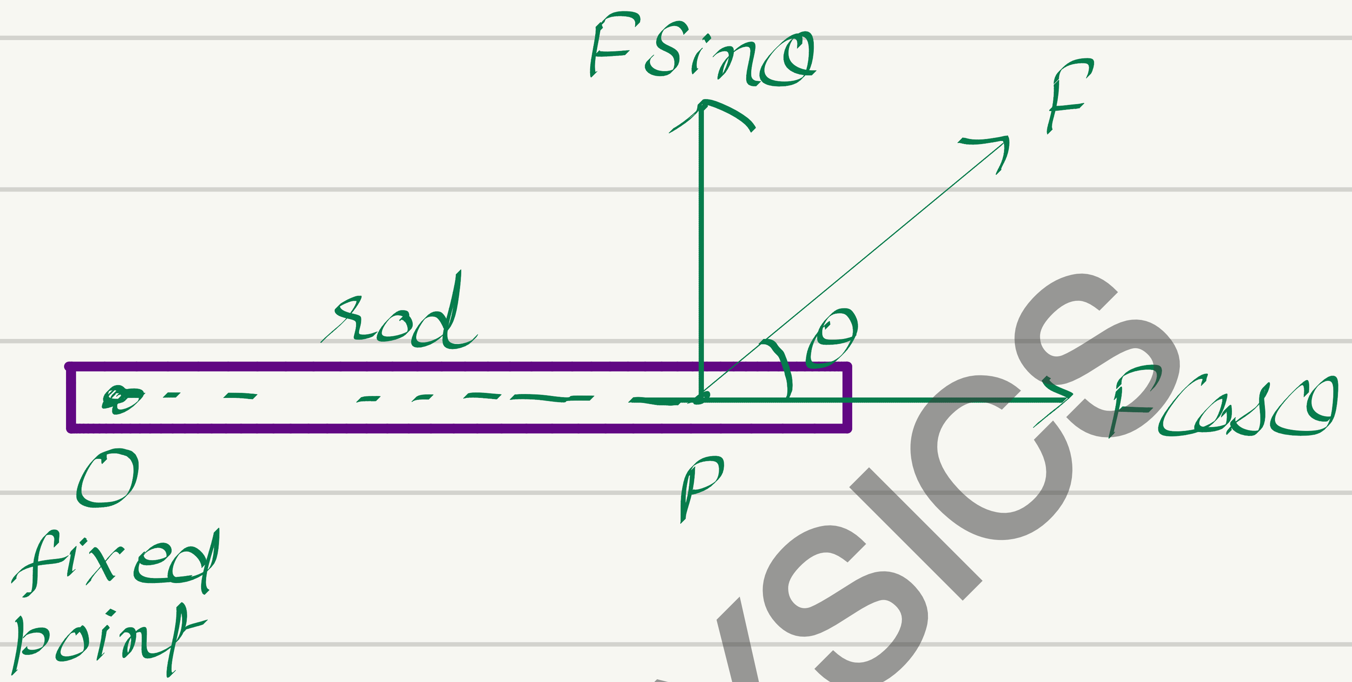
$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

or $$\boxed{\tau = rF\sin\theta}$$

Moment of force is known as torque

Torque = Force × ⊥ distance of the line of action of force from the axis of rotation.

* Torque is a twisting force, causes rotation.



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Unit - N-m (same as work but cannot be written joule)

Dimⁿ - $[ML^2T^{-2}]$

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

OQ is ⊥ distance from axis of rotation

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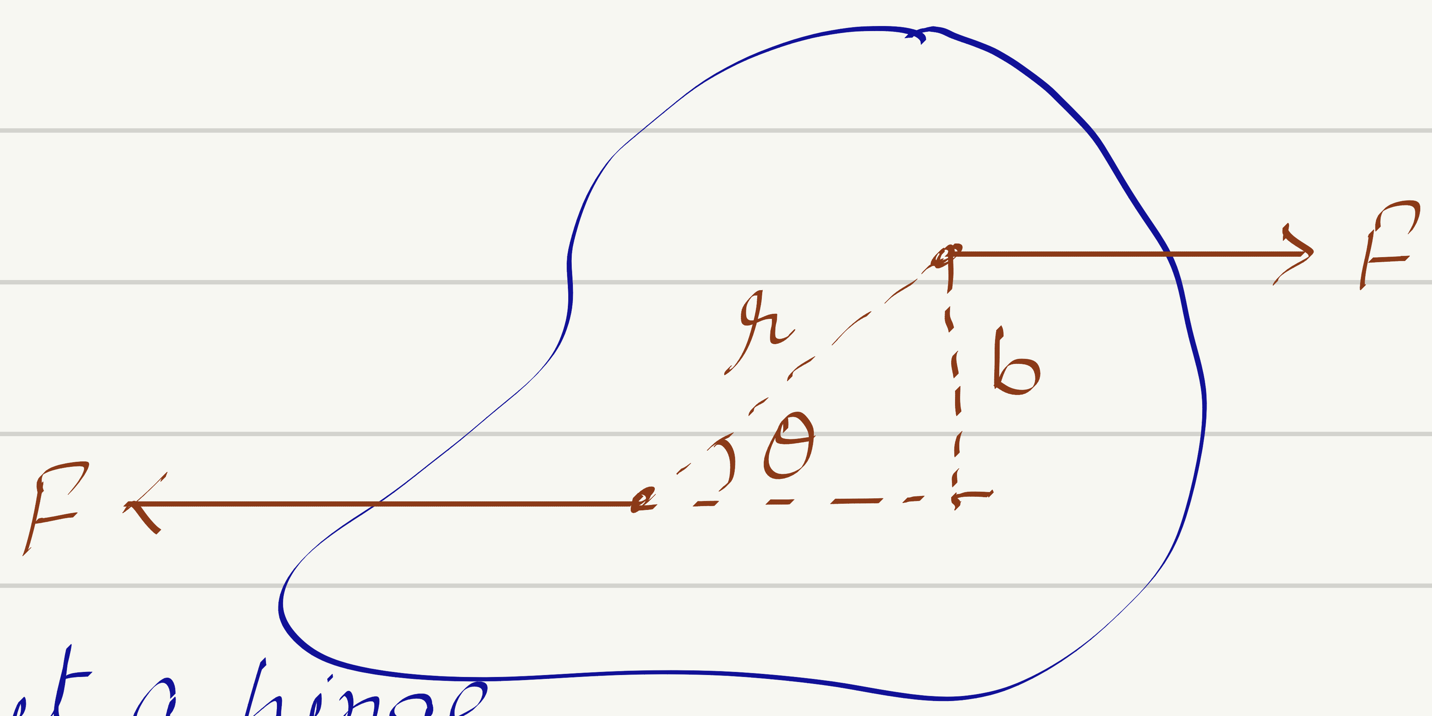
__Couple of forces:__ When two forces of equal magnitude act on different points and in opposite direction with different lines of action these forces form a couple.

$\tau = Fr\sin\theta$

or $\tau = Fb$



* Rotation of a door about a hinge, rotation of griding wheel, unbolting a nut by wrench are examples of involving torque.

$\tau = \text{constant} \Rightarrow Fr\sin\theta = \text{constant}$

or $F = \dfrac{\text{constant}}{r\sin\theta} \Rightarrow F \propto \dfrac{1}{r}$ and $F \propto \dfrac{1}{\sin\theta}$

Thus, longer the arm and greater the value of $\sin\theta$, lesser will be the force required.
That's why wrench with longer handles are more useful than the shorter one.

* Torque is an axial vector. Its dir^n is always perpendicular to the plane containing $\vec{r}$ and $\vec{F}$.
* Its dir^n is determined by the right hand thumb rule.
* Generally, anticlockwise torques are taken +ve and clockwise torques are taken -ve.

Rotational Equilibrium:

A rigid body is said to be in a state of rotational equilibrium if its angular acceleration is zero.
i.e for rotational equilibrium the body must be in rest or in rotation with a constant angular velocity.
In rotational equilibrium

$$\boxed{\tau_{net} = 0} \qquad [\alpha = 0]$$

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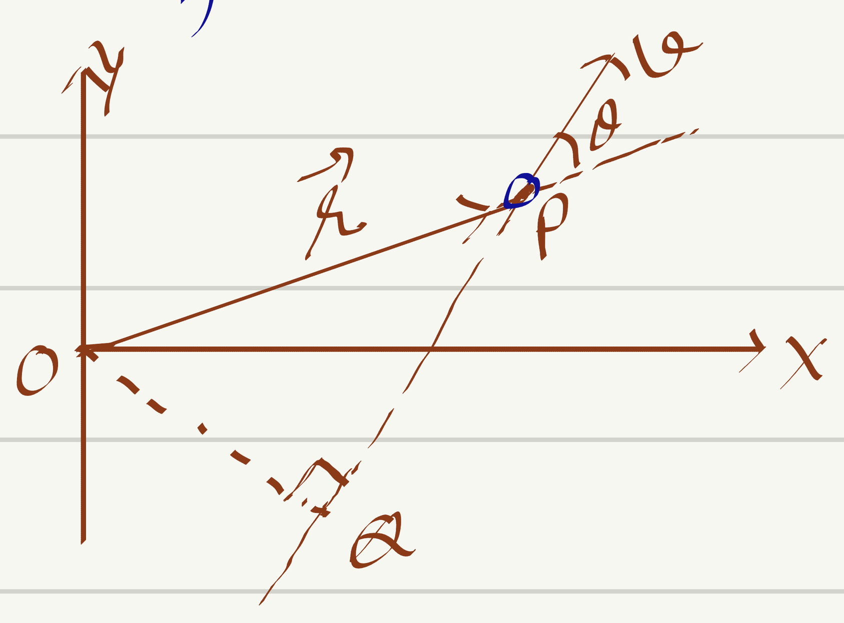
Angular Momentum (L):

Angular momentum of a particle - It is defined as the moment of the linear momentum.

$$\boxed{\vec{L} = \vec{r} \times \vec{p}}$$

$$\vec{L} = \vec{r} \times m\vec{v} \qquad [\because p = mv]$$

$$\boxed{L = mvr\sin\theta}$$



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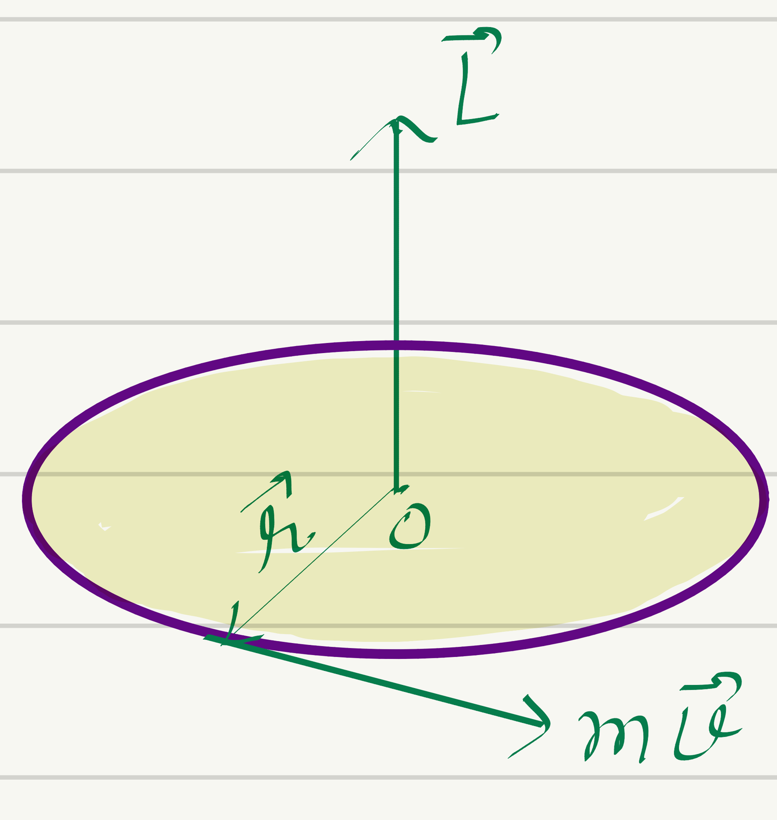
Angular momentum of a rigid body - It is measure of the quantity of rotational motion in a body.

Angular momentum about a fixed axis

$$L = m_1v_1r_1 + m_2v_2r_2 + \dots$$
$$= m_1(r_1\omega)r_1 + m_2(r_2\omega)r_2 + \dots$$
$$= m_1r_1^2\omega + m_2r_2^2\omega + \dots$$
$$= (m_1r_1^2 + m_2r_2^2 + \dots)\omega$$
$$\boxed{L = I\omega} \qquad \left[\because I = \sum_{i=1}^{n} m_ir_i^2\right]$$



$\vec{L}$ is a vector quantity, dirⁿ is same as dirⁿ of $\vec{\omega}$

Unit - $kg\, m^2 s^{-1}$

dimⁿ - $[M L^2 T^{-1}]$

Relation between torque and angular momentum (τ & L)

For a rotating body

$$\vec{L} = \vec{r} \times \vec{p}$$

on differentiating w.r. to 't'

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p})$$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} + \frac{d\vec{r}}{dt} \times \vec{p}$$

$$= \vec{r} \times \vec{F} + \vec{v} \times \vec{p}$$

$$= \vec{r} \times \vec{F} + \vec{v} \times (m\vec{v})$$

$$= \vec{r} \times \vec{F} + 0 \qquad [\vec{v} \times \vec{v} = 0, \text{ as } \sin 0° = 0]$$

so $\dfrac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$

or $\boxed{\dfrac{d\vec{L}}{dt} = \vec{\tau}}$

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So the time rate of angular momentum is equal to torque.

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Relation between torque and angular acceleration (τ & α)

We know

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau} = \frac{d}{dt}(I\vec{\omega}) \qquad [L = I\omega]$$

$$= I\frac{d\vec{\omega}}{dt}$$

$$\boxed{\vec{\tau} = I\vec{\alpha}}$$

Relation b/w τ and L

$$L = I\omega$$

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

$$= I\alpha \qquad [\because \alpha = \frac{d\omega}{dt}]$$

$$\boxed{\frac{dL}{dt} = \tau} \qquad [\because \tau = I\alpha]$$

* Angular momentum is an axial vector.
* In cartesean coordinates -

$$\vec{L} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$
$$= m[(x\hat{i} + y\hat{j} + z\hat{k}) \times (v_x\hat{i} + v_y\hat{j} + v_z\hat{k})]$$

i.e $\vec{L} = m[\hat{i}(yv_z - zv_y) - \hat{j}(xv_z - zv_x) + \hat{k}(xv_y - yv_x)]$

* By $L = mvr \sin\theta$

(i) for $\theta = 0°$ or $180°$ i.e $\vec{r}$ and $\vec{v}$ are parallel and antill

$L = 0$ $[\sin 0° = 0]$

(b) for $\theta = 90°$

$L = mvr$ [L is max^m]

i.e in case of circular motion L is max^m and is mvr.

* Direction of L is given by right hand thumb rule.

Conservation of angular momentum:

The angular momentum of a body is conserved if the resultant external torque on the body is zero.

Proof - Consider a rigid body rotating with α angular acceleration. We know

$$\vec{L} = I\vec{\omega}$$

$$\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} \qquad [I \text{ constant}]$$

$$\frac{d\vec{L}}{dt} = I\vec{\alpha}$$

or $\frac{d\vec{L}}{dt} = \vec{\tau}_{ext}$ [$\because$ resultant external torque $\vec{\tau} = I\vec{\alpha}$]

hence if $\vec{\tau}_{ext} = 0$

$$\frac{dL}{dt} = 0$$

i.e $\boxed{L = \text{Constant}}$

or $I\omega = \text{Constant}$

$$\boxed{I_1\omega_1 = I_2\omega_2}$$

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Centre of Gravity: The centre of gravity is that point of the body, where the whole weight of the body is supposed to be concentrated.

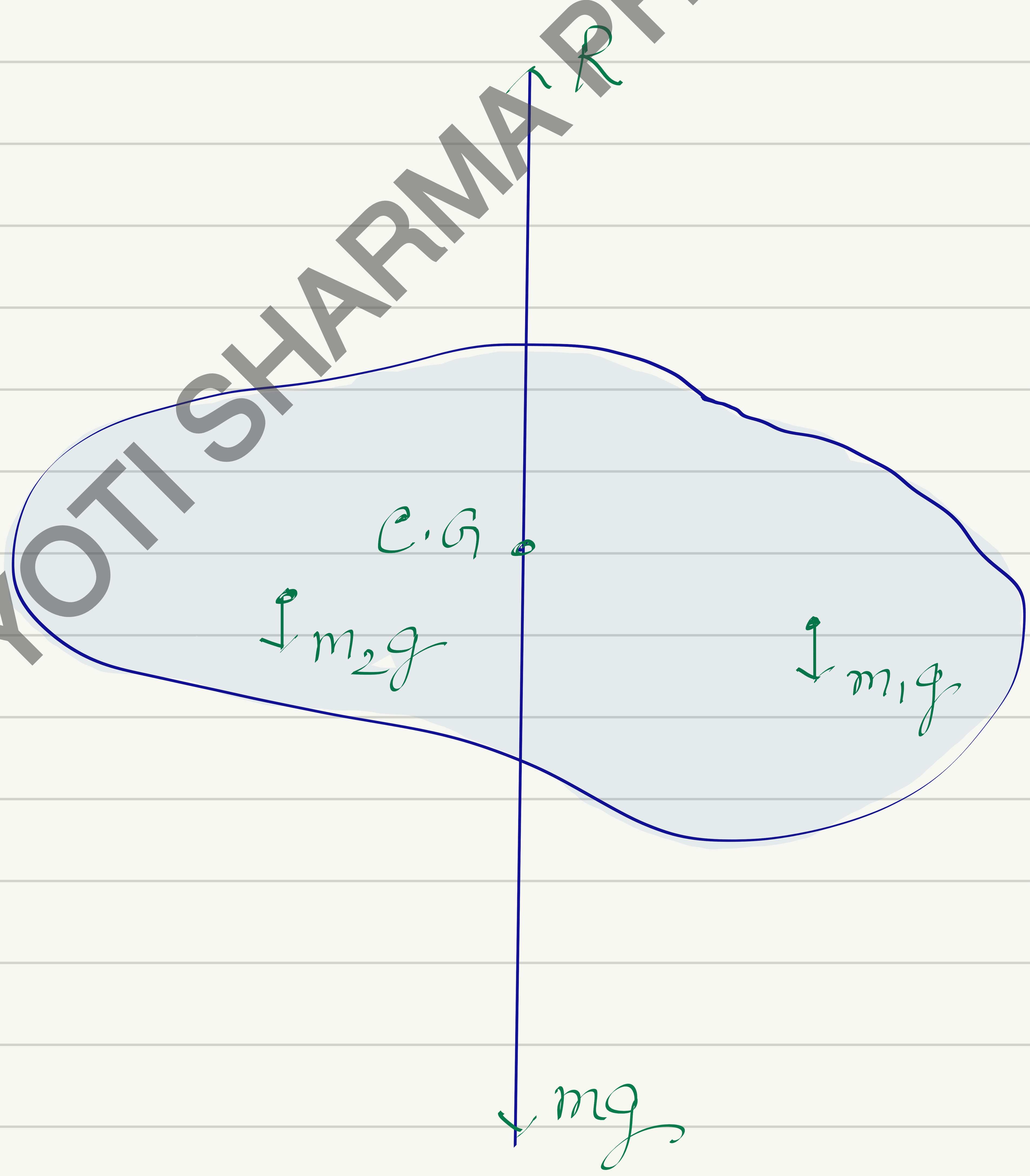
The total torque about the centre of gravity is zero.

$$\sum_{i=1}^{n} \vec{\tau}_i = \sum_{i=1}^{n} \vec{r}_i \times \vec{F}_i = \sum \vec{r}_i \times m_i \vec{g} = 0$$

i.e

$$\sum \tau = 0$$

Therefore we may define the centre of gravity as.
It is a point of a body where the total gravitational torque acting on the body is zero.



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* Centre of gravity coincides with the centre of mass.
* Centre of mass does not depend on gravity it depends only on the mass distribution of the body.

Equilibrium of a rigid body: (Mechanical Equilibrium)

A body is said to be in mechanical equilibrium if its linear momentum and angular momentum are not changing with time.
i.e no linear accⁿ and no angular accⁿ.

(i) Translational equilibrium

If vector sum of the forces acting on a rigid body is zero then the body is said to be in translational equilibrium. i.e

$$\Sigma F_{ext} = 0 \quad \Rightarrow \text{Translational equilibrium}$$

(ii) Rotational equilibrium

If the vector sum of the torques on the rigid body is zero the body is said to be in rotational equilibrium. i.e

$$\Sigma \tau_{ext} = 0 \quad \Rightarrow \text{Rotational equilibrium}$$

Thus, If $\Sigma F_{ext} = 0$ and $\Sigma \tau_{ext} = 0$ the rigid body is in Mechanical Equilibrium.

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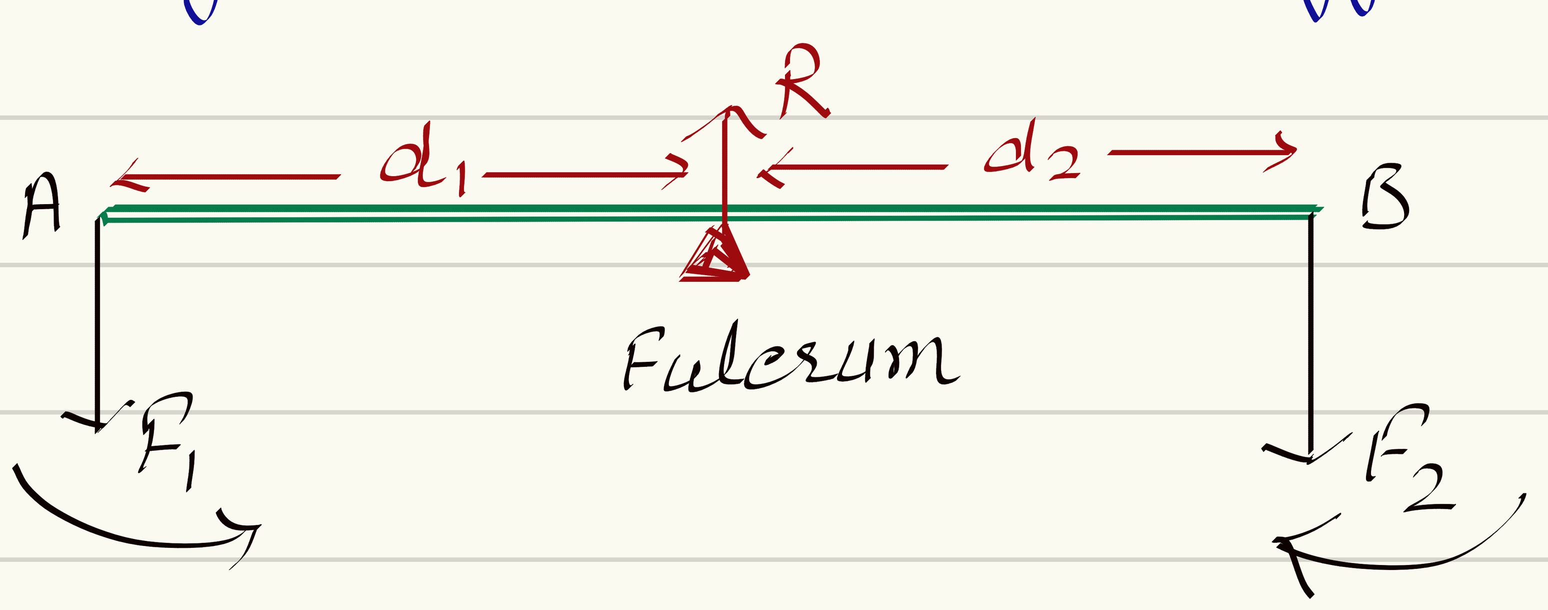
Principle of Moments - The lever is system in mechanical equilibrium. e.g. see-saw in playground.



F_1 and F_2 are two parallel forces acting at distance d_1 and d_2 from the fulcrum.
R is the reaction force of the support at fulcrum.
For translation equilibrium

$$\Sigma F_{ext} = 0$$

$$\text{i.e} \quad R - F_1 - F_2 = 0 \quad — (1)$$

For rotational equilibrium

$$\Sigma \tau_{ext} = 0$$

τ_1 of $F_1 = d_1 F_1$ [anticlockwise (+ve)]

τ_2 of $F_2 = d_2 F_2$ [clockwise (-ve)]

and $d_1F_1 - d_2F_2 = 0$ — (ii) [for rotational equilibrium]

or $\boxed{d_1F_1 = d_2F_2}$

* $\frac{F_1}{F_2}$ is called mechanical advantage (M.A)

$$\boxed{\frac{F_1}{F_2} = \frac{d_2}{d_1}}$$

Kinematics of Rotational motion -

Kinematics equations for rotational motion

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

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$$\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$$

Where $\theta_0 \rightarrow$ initial angular displacement, $\omega_0 \rightarrow$ initial angular velocit

* (Derivation of these equations is similar as derivation of kinematic equations in uniformly accelerated translation motion)

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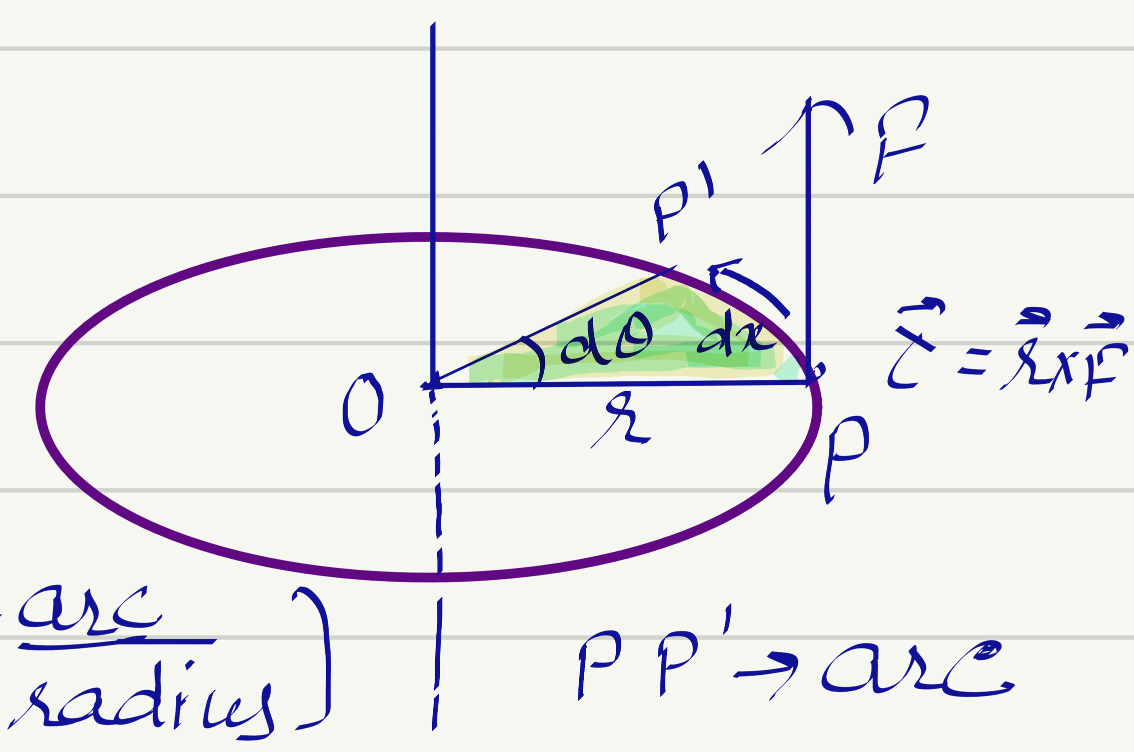
Work done by a torque:

In rotional motion



$$\vec{\tau} = \vec{r} \times \vec{F}$$

From fig. $d\theta = \frac{PP'}{r} = \frac{dx}{r}$ [angle = $\frac{\text{arc}}{\text{radius}}$] | $PP' \rightarrow$ arc

or $dx = r\,d\theta$

Now $dW = F \cdot dx \cos\theta$

for dx displacement, $\theta = 0$, then

$$dW = F\,dx$$

$$= F(r\,d\theta) \qquad [\because dx = r\,d\theta]$$

$$= F\,r\,d\theta$$

$$\boxed{dW = \tau\,d\theta} \qquad [\because \tau = Fr]$$

This is the work done by the total external torque τ which acts on the body rotating about a fix axis.

Rotational Power: Power is the rate of doing work.

i.e. $$P = \frac{dW}{dt}$$

$$= \frac{d}{dt}(\tau\, d\theta)$$

$$= \tau \frac{d\theta}{dt}$$

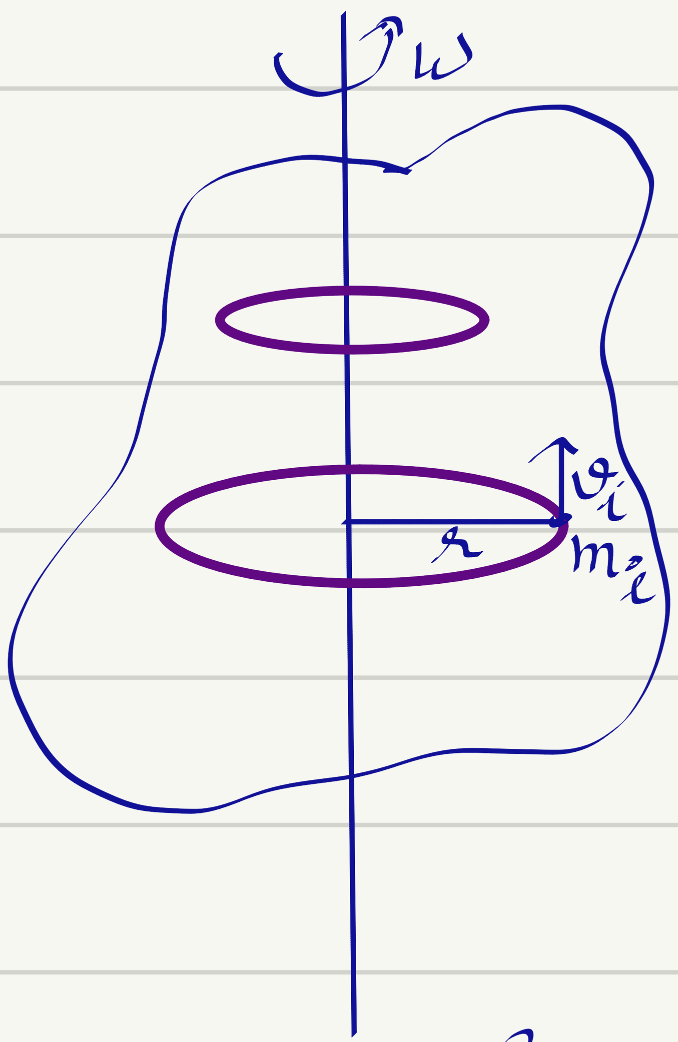
$$\boxed{P = \tau\omega} \qquad \left[\because \omega = \frac{d\theta}{dt}\right]$$

This is the instantaneous power.

Rotational Kinetic Energy: The energy possessed due to rotational motion of a body is known as rotational kinetic energy.

A rigid body of system of n particles having masses $m_1, m_2 --- m_n$ is rotating about an axis with uniform velocity ω.

The total kinetic energy of the body is



$$K.E_r = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 + --- \frac{1}{2} m_n v_n^2$$

$$= \frac{1}{2} m_1 (r_1\omega)^2 + \frac{1}{2} m_2 (r_2\omega)^2 + --- \frac{1}{2} m_n (r_n\omega)^2$$

$$= \frac{1}{2} m_1 r^2\omega^2 + \frac{1}{2} m_2 r_2^2\omega^2 + --- \frac{1}{2} m_n r_n^2 \omega^2$$

$$\boxed{K.E_r = \frac{1}{2} I\omega^2} \qquad \left[I = \sum_{i=1}^{n} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + -- m_n r_n^2\right]$$

* It is scalar quantity

* $$\boxed{K.E_r = \frac{1}{2} I\omega^2 = \frac{1}{2} MK^2\omega^2}$$

$$\boxed{K.E_r = \frac{1}{2}(I\omega)\omega = \frac{L\omega}{2} = \frac{L}{2}\left(\frac{L}{I}\right) = \frac{L^2}{2I}}$$

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Rolling Motion:

The combined motion of rotatory motion and translation motion is called rolling motion. The velocity of COM represents linear motion while angular velocity represents rotatory motion.

Total energy in rolling = Translatory K.E + Rotatory K.E
$= K.E + K.E_r$

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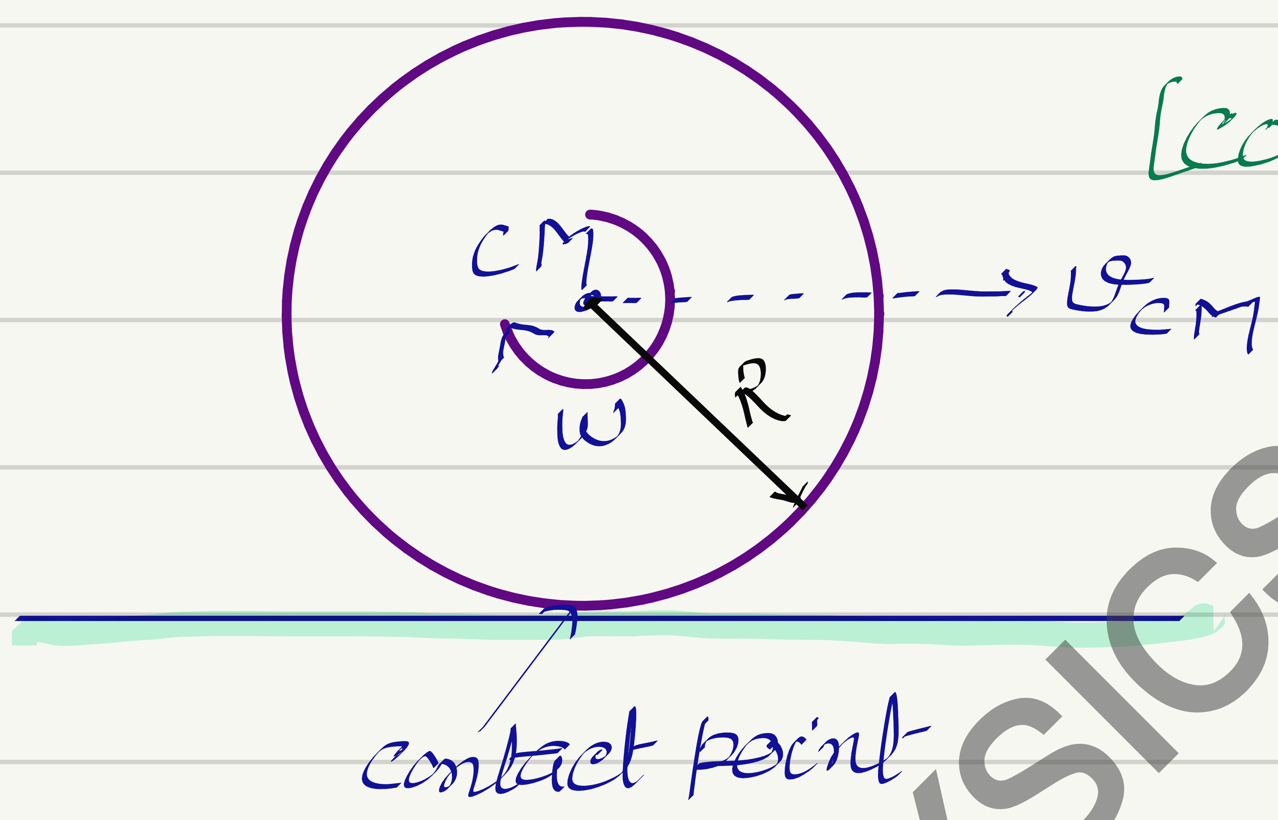
[COM/CM → centre of mass]



$v_{cm} = R\omega$

* Rolling without slipping (Pure rolling motion)

$$v_{cm} = R\omega$$

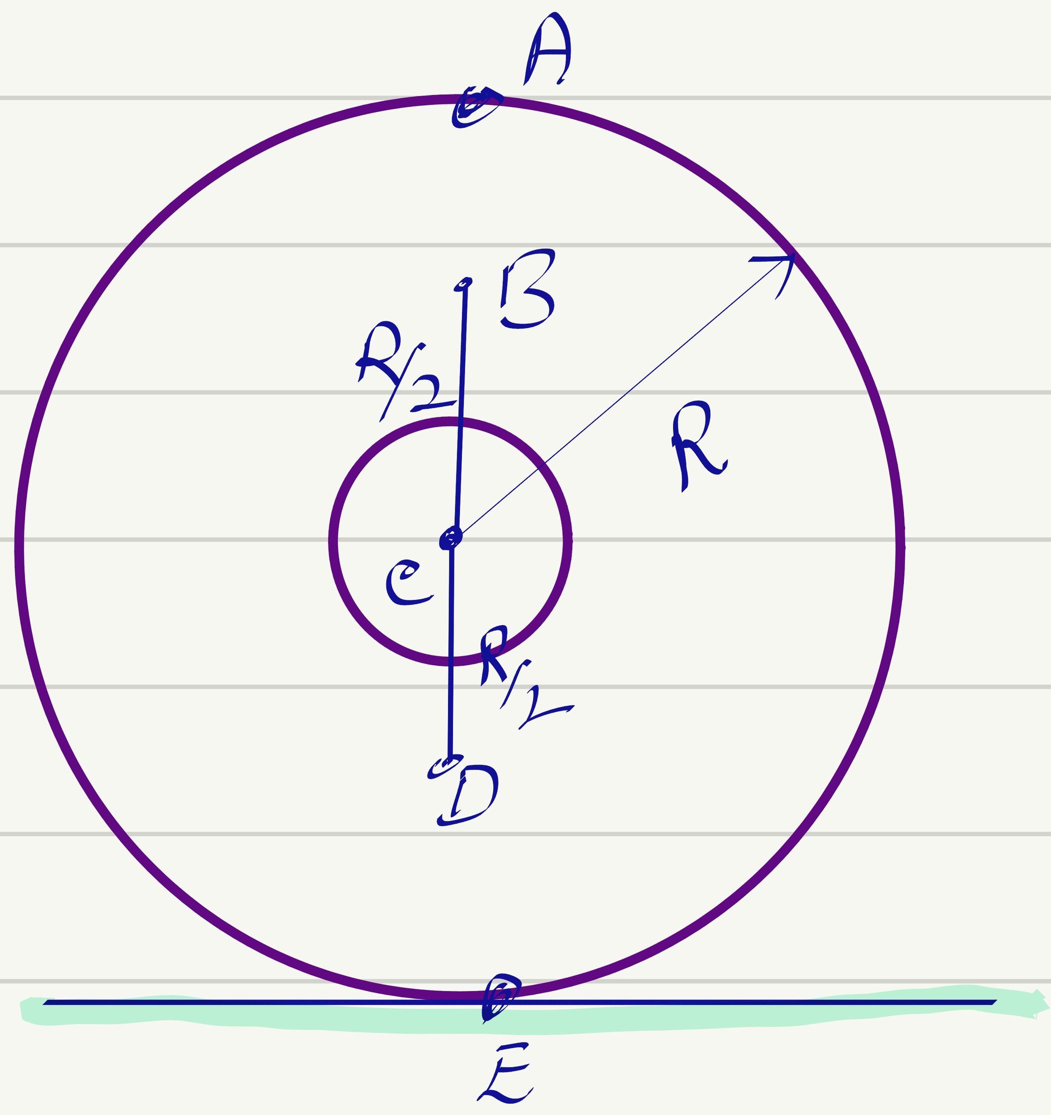
* In pure rolling velocity at any point of the body with respect to surface $\vec{v} = \vec{v}_{cm} + \vec{\omega} \times \vec{r}$

(I) At 'A'

$v_A = v_{cm} + \omega R$

$= v_{cm} + v_{cm}$

$v_A = 2v_{cm}$



(II) At 'B'

$v_B = v_{cm} + \frac{\omega R}{2} = v_D$

$= v_{cm} + \frac{v_{cm}}{2}$

$v_B = \frac{3}{2} v_{cm}$

(III) At 'D'

$v_D = v_{cm} + \frac{\omega R}{4}$

$v_D = \frac{3}{4} v_{cm}$

(IV) At 'E'

$v_E = v_{cm} - R\omega = v_{cm} - v_{cm} = 0$

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Rotational Kinetic Energy:

Rolling Kinetic Energy

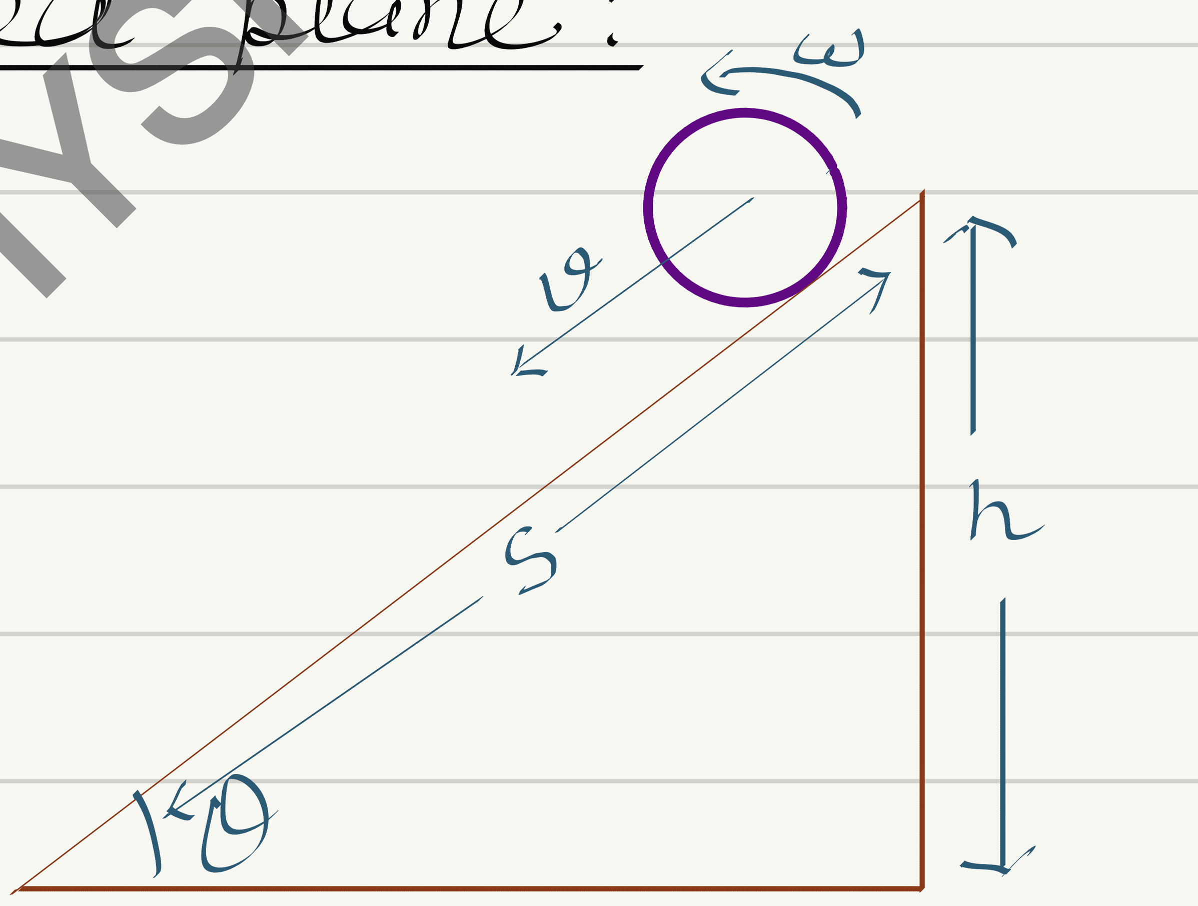
$$E = \frac{1}{2} m v^2 + \frac{1}{2} I \omega^2$$

$$= \frac{1}{2} m v^2 + \frac{1}{2} m K^2 \left(\frac{v^2}{R^2}\right) \qquad [v = r\omega]$$

$$= \frac{1}{2} m v^2 \left(1 + \frac{K^2}{R^2}\right)$$

* $E_{translational} : E_{rotation} : E_{Total} = 1 : \frac{K^2}{R^2} : 1 + \frac{K^2}{R^2}$

Rolling motion on an inclined plane:



$h = s \sin\theta$

* Velocity at the bottom

$$mgh = \frac{1}{2} m v^2 \left(1 + \frac{K^2}{R^2}\right) \quad \text{— (1)}$$

and $h = s \sin\theta$ — (2)

from (1) & (2)

$$\cancel{m} g \times s \sin\theta = \frac{1}{2} \cancel{m} v^2 \left(1 + \frac{K^2}{R^2}\right)$$

or $$v = \sqrt{\frac{2 g s \sin\theta}{\left(1 + \frac{K^2}{R^2}\right)}} = \sqrt{\frac{2gh}{\left(1 + \frac{K^2}{R^2}\right)}}$$

* If different bodies are allowed to roll down - least $\frac{K^2}{R^2}$ will reach first

* When a ring, disc, hollow sphere and a solid sphere of same R roll on the same inclined plane -

$$v_S > v_D > v_H > v_R$$

as K is least for solid sphere.

S → solid sphere
D → Disc
H → hollow sphere
R → ring

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Body	$\frac{K^2}{R^2}$	$\frac{E_{trans}}{E_{rotation}} = \frac{1}{\left(\frac{K^2}{R^2}\right)}$	$\frac{E_{trans}}{E_{total}} = \frac{1}{\left(1+\frac{K^2}{R^2}\right)}$	$\frac{E_{rotation}}{E_{total}} = \frac{K^2/R^2}{\left(1+\frac{K^2}{R^2}\right)}$
Ring	1	1	$\frac{1}{2}$	$\frac{1}{2}$
Disc	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Solid sphere	$\frac{2}{5}$	$\frac{5}{2}$	$\frac{5}{7}$	$\frac{2}{7}$
Spherical shell	$\frac{2}{3}$	$\frac{3}{2}$	$\frac{3}{5}$	$\frac{2}{5}$
Solid cylinder	$\frac{1}{2}$	2	$\frac{2}{3}$	$\frac{1}{3}$
Hollow cylinder	1	1	$\frac{1}{2}$	$\frac{1}{2}$

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