GRAVITATION

CHAPTER - 7

Kepler's Law of Planetary Motion

The three law of kepler -

1. **Law of Orbits** :- All planets move in elliptical orbits with sun situated a one of the foci of the ellipse.

2. **Law of Areas** : The line that joins any planet to the sun sweeps equal areas in equal interval of time. i.e

$$\frac{\Delta A}{\Delta t} = \text{constant}$$

3. **Law of Periods** :-

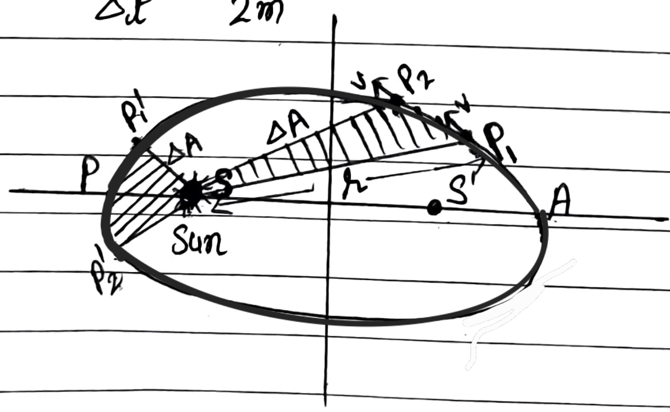
The square of the time period of revolution of a planet is proportional to the cube of the semi-major axis of the ellipse. i.e

$$T^2 \propto r^3$$

or $\boxed{T^2 = K r^3}$ $K \rightarrow$ constant

* Planets move slower when they are farther from sun and move faster when they are nearer to the sun.

* $\frac{\Delta A}{\Delta t} = \frac{L}{2m} = \text{Constant}$, $L \rightarrow$ Angular momentum



P → Perihelion

A → Aphelion

Universal law of Gravitation -

Mathematical Form -

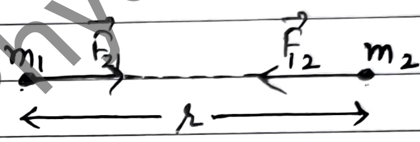
If m_1 and m_2 are the masses of two particles and r is the distance between them, the force of attraction between them has magnitude

$$\boxed{F = \frac{G m_1 m_2}{r^2}}$$

where G is the universal constant of gravitation.

$G = 6.67 \times 10^{-11}\ N m^2/Kg^2$, $[G] = [M^{-1} L^3 T^{-2}]$

Vector Form -



$$\vec{F}_{21} = \frac{G m_1 m_2}{r^2} \hat{r}_{12}$$

and $$\vec{F}_{12} = \frac{G m_1 m_2}{r^2} \hat{r}_{21}$$

but $\vec{r}_{12} = -\vec{r}_{21}$, so

$$\boxed{\vec{F}_{12} = -\vec{F}_{21}}$$

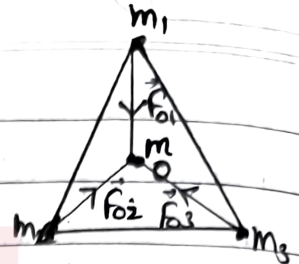
Characteristic of Gravitational force -

(i) It is central force.
(ii) It is independent of intervening medium.
(iii) It does not depent on the presence of other bodies.
(iv) It is valid for sperically symmetrical point objects.
(v) It is extremely small force.

Principle of Superposition of Gravitation.

The resultant gravitational force $\vec{F}$, acting on a particle due to a number of point masses is the vector sum of the forces due to all individual masses on the particle.

$$\vec{F}_{net} = \vec{F}_{o1} + \vec{F}_{o2} + \vec{F}_{o3} + \cdots$$



Acceleration due to Gravity (g):-

The acceleration produced in a body due

If a body of mass 'm' lying on earth's surface the gravitational force is given by

$$F = \frac{GMm}{R^2}$$

$M \rightarrow$ Mass of earth
$m \rightarrow$ mass of the body
$R \rightarrow$ Radius of the earth

put $F = mg$, then

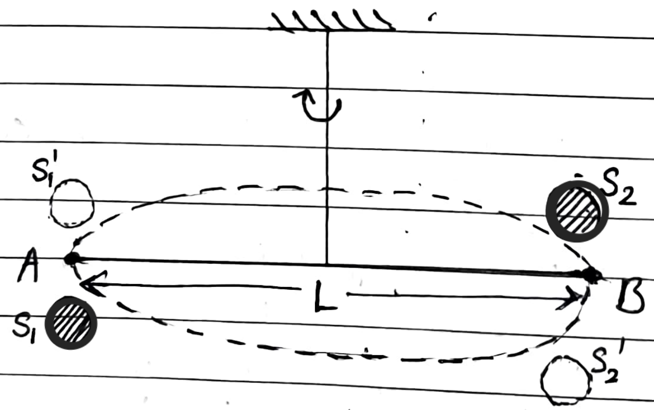
$$mg = \frac{GMm}{R^2}$$

or $\boxed{g = \frac{GM}{R^2}}$ $\qquad g = 9.8\ m/s^2$

* At moon value of g is one sixth of it value at earth. $\boxed{g_m = \frac{g_e}{6}}$

The Gravitational Constant: Cavendish's Experiment

The value of 'G' can be determined experimentally by cavendish Experiment.



Here

$$F = \frac{GMm}{d^2}$$

At the equilibrium,

$$\frac{GMm}{d^2} L = \tau\theta$$

Observation of θ thus enables the caculation of G. The currently accepted value of $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

Mass and Density of Earth

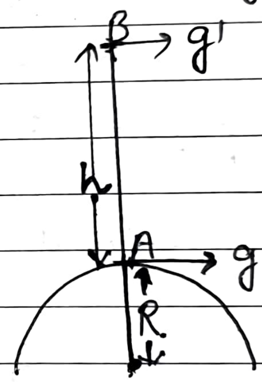
Mass of earth $M = \dfrac{gR^2}{G}$

Mean density of the earth

$$\rho = \frac{3g}{4\pi G R} \quad \left[\because \rho = \frac{M}{V}\right]$$

Variation of Acceleration Due to Gravity

(a) Effect of Altitude:



Earth

R → Radius of earth

At the surface of earth (At 'A')

$$g = \frac{GM}{R^2} \quad -(1)$$

At h height from earth surface
At 'B'.

$$g' = \frac{GM}{(R+h)^2} \quad -(2)$$

from (1) and (2)

$$\frac{g'}{g} = \frac{GM/(R+h)^2}{GM/R^2}$$

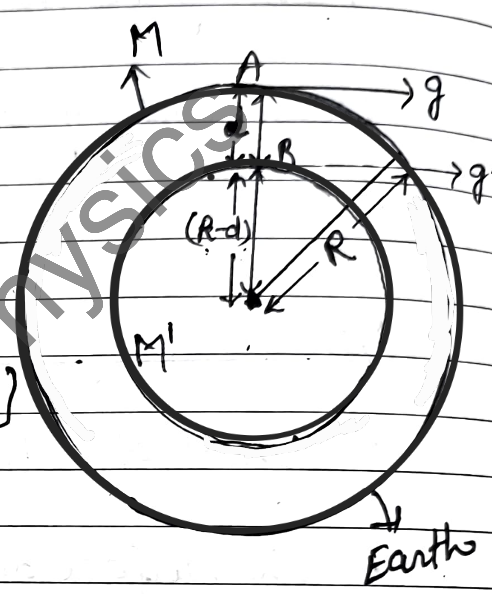
$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{R^2\left(1+\frac{h}{R}\right)^2}$$

$$\frac{g'}{g} = \left(1+\frac{h}{R}\right)^{-2}$$

Using binomial expression for $\frac{h}{R} << 1$

$$\boxed{g' = g\left[1 - \frac{2h}{R}\right]} \qquad g' < g$$

i.e for small height h value of g decreases by a factor $\left(1 - \frac{2h}{R}\right)$.



Effect of Depth :

At the surface of Earth (At 'A')

$$g = \frac{GM}{R^2}$$

$$= \frac{G}{R^2}\left(\frac{4}{3}\pi R^3 \cdot \rho\right) \qquad \{M = V \cdot \rho\}$$

$$g = \frac{4}{3} G \pi R \rho \quad \text{— (1)}$$

At depth d (At B)

$$g' = \frac{GM'}{(R-d)^2} = \frac{G \cdot \frac{4}{3}\pi (R-d)^3 \cdot \rho}{(R-d)^2}$$

or $$g' = \frac{4}{3} G \pi (R-d)\rho \quad \text{— (2)}$$

from (1) and (2)

$$\frac{g'}{g} = \frac{\frac{4}{3} G\pi (R-d)\rho}{\frac{4}{3} G\pi R \rho}$$

or $$\frac{g'}{g} = \frac{R-d}{R}$$

or $$\boxed{g' = g\left(1 - \frac{d}{R}\right)} \qquad g' < g$$

Thus as we go down below the earth surface the value of g decreases by a factor $\left(1-\frac{d}{R}\right)$.

* At the centre of the earth $d=R$, then

$$g'=0$$

i.e at the centre of the earth g becomes zero.

* Value of 'g' is maximum at the surface of earth and it decreases whether we go up or down.

Gravitational field:

The space around a body in which any other body experiences a gravitational force is called gravitational field.

Intensity of Gravitational field

Intensity of gravitational field at any point is given by

$$I = \frac{F}{m} = \frac{\frac{GMm}{r^2}}{m}$$

$$\boxed{I = \frac{GM}{r^2}}$$

unit - N/Kg Dimension of $I = [L^1T^{-2}]$

Here we can see that

$I = g$ (acc^n due to gravit) $\left[\because \frac{GM}{r^2} = g\right]$

i.e The gravitational field intensity due to earth is called acc^n due to gravity.

Gravitational Potential -

It is defined as the amount of work done in bringing a body of unit mass from infinity to a point inside the field.

$$V = -\frac{GM}{r}$$

Unit - J/Kg

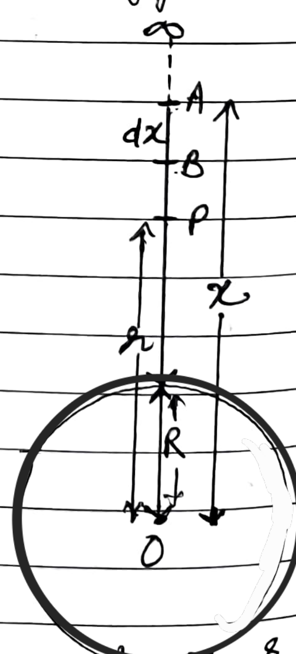
Gravitational Potential energy

It is equal to the work done in bringing an object from infinity to a point in the gravitational field of a body.

$$U = -\frac{GMm}{r}$$

-ve sign shows that the potential energy U decreases with increase in distance.

Expression for Gravitational Potential Energy



Let OP = r and r > R

When the body is at A

$$F = \frac{GMm}{x^2}$$

the work done in moving a distance dx

$$dW = F\,dx$$

$$= \frac{GMm}{x^2} \cdot dx$$

the total work done in moving the body from ∞ to P. i.e from $x = \infty$ to $x = r$

$$W = \int dW$$

$$= \int_{\infty}^{r} \frac{GMm}{x^2} \cdot dx$$

$$= GMm \int_{\infty}^{r} x^{-2} dx \qquad \int x^n dx = \frac{x^{n+1}}{n+1}$$

$$= -GMm \left[\frac{1}{x}\right]_{\infty}^{r}$$

$$W = -GMm \left[\frac{1}{r} - \frac{1}{\infty}\right]$$

or $$W = -\frac{GMm}{r}$$

This work done is the gravitational Potential Energy

i.e $$\boxed{U = -\frac{GMm}{r}}$$

* U is maximum at ∞. At ∞ its value is zero.
* For an isolated system of particles the total potential energy is equal to the sum of energies for all possible pairs.

Escape Speed -

The minimum speed required by an object to escape from the gravitational field of earth and never returns.

Escape speed $V_e = \sqrt{\frac{2GM}{R}}$

For earth $V_e = 11.2$ Km/s

Expression for the Escape speed :-

By the conservation of energy -

$$(K + U)_i = (K + U)_f$$

where

$$K = \frac{1}{2} m v^2$$

and

$$U = -\frac{GMm}{r}$$

Here U_f and K_f are zero because at ∞ U becomes zero and K_f will also be zero because final velocity will be zero.

Thus,

$$\frac{1}{2} m v_e^2 - \frac{GMm}{r} = 0 + 0$$

or

$$\frac{1}{2} \cancel{m} v_e^2 = \frac{GM\cancel{m}}{r}$$

or

$$v_e^2 = \frac{2GM}{r}$$

or

$$\boxed{v_e = \sqrt{\frac{2GM}{r}}}$$

but $GM = gr^2$, then

$$\boxed{v_e = \sqrt{2gr}}$$

* For earth escape speed is 11.2 Km/s
* For moon escape speed is 2.3 Km/s, which is 5 times smaller than earth. This is the reason moon has no atmosphere.

Earth Satellites -

Earth satellites are the objects which revolve around the earth.

Orbital Speed of a Satellite:

The minimum speed required to put the satellite into a given orbit around the earth.

We consider a satellite in a circular orbit of distance $(R+h)$. Where R is radius of earth and h is the height of satellite from earth surface.

Here the centripetal force is provided by the gravitational force. So

$$\frac{mV_0^2}{(R+h)} = \frac{GMm}{(R+h)^2}$$

or $$V_0^2 = \frac{GM}{R+h}$$

or $$V_0 = \sqrt{\frac{GM}{R+h}}$$

Thus V_0 decreases as h increases.

For $h = 0$

$$V_0 = \sqrt{\frac{GM}{R}}$$

$$V_0 = \sqrt{gR} \qquad [\because GM = gR^2]$$

For earth $V_0 \simeq 7.8$ km/s [$\because$ for low earth orbit]

Time Period of a Satellite -

$$T = \frac{2\pi(R+h)}{V_0}$$

here $V_0 = \sqrt{\frac{GM}{R+h}}$

so $$T = \frac{2\pi(R+h)}{\sqrt{\frac{GM}{R+h}}}$$

$$T = \frac{2\pi(R+h)^{3/2}}{\sqrt{GM}} \quad \text{—(1)}$$

On squaring the both sides

$$T^2 = \frac{4\pi^2(R+h)^3}{GM}$$

$$T^2 = K(R+h)^3 \quad \text{—(2)}$$

where $K = \frac{4\pi^2}{GM}$

$T^2 = K(R+h)^3$ verifies Kepler's law of period.

* If satellite is very close to the earth i.e. h is very small then

$$T_0 = \frac{2\pi R^{3/2}}{\sqrt{GM}} \quad \text{[from eq}^n\text{(1)]}$$

or $$T_0 = \frac{2\pi R^{3/2}}{\sqrt{gR^2}} \quad [\because GM = gR^2]$$

or $$\boxed{T_0 = 2\pi\sqrt{\frac{R}{g}}}$$

put $R = 6400\,km$, $g = 9.8\,m/s^2$
we get

$$T_0 \approx 85\ min$$

Energy of An Orbiting Satellite -

$$K.E = \frac{1}{2} m v_0^2$$

$$= \frac{1}{2} m \frac{GM}{R+h} \qquad \left[\because v_0 = \sqrt{\frac{GM}{R+h}}\right]$$

$$K.E = \frac{GMm}{2(R+h)} \quad \text{——(1)}$$

Now the gravitational potential energy at a distance $(R+h)$ is

$$P.E = -\frac{GMm}{R+h} \quad \text{——(2)}$$

Total Energy E

$$E = K.E + P.E$$

$$= \frac{GMm}{2(R+h)} - \frac{GMm}{R+h}$$

$$\text{Total Energy } E = -\frac{GMm}{2(R+h)} \quad \text{——(3)}$$

* E is also called Binding Energy of the Satellite.
* From (1), (2) and (3)

$$-E = +K.E = -\frac{P.E}{2}$$

* In the circular orbit case the total energy E is constant and -ve.
* Satellites are always at a finite distance from earth and hence their energies can not be positive or zero.

Geostationary and Polar Satellites-

Geostationary Satellite -

The satellite which always appears stationary to an observer on the earth is called geostationary satellite or geosynchronous satellite.

* They move around the earth in equatorial plane.
* They have same time period as the rotation of earth.
* High altitude satellite ($h \simeq 36{,}000$ km)

Uses - Radio communication, TV transmission

Polar Satellite :-

The satellites revolve around the earth in a circles north to south orbit, passes over poles.

* Low Altitude ($h \simeq 500 - 800$ Km)
* Angles of inclinations of orbit with equatorial plane is $90°$.
* Also called sun synchronous satellite

Uses - Monitering climate change, Remote sensing, meteorology and environment studies.

Weighlessness-

It is the situation in which effective weight of the body becomes zero.
An astronaute experiences a weightlessness in a space satellite.

* Gravitational field due to hollow sphere

(i) If $x < R$ i.e. inside the sphere
gravitational field $\boxed{I = 0}$

(ii) If $x = R$ i.e at the surface

$$I = \frac{-GM}{R^2}$$

(iii) If $x > R$ i.e outside the sphere

$$I = \frac{-GM}{x^2}$$

* Gravitational field due to solid sphere

(i) If $x < R$

$$I = \frac{-GMx}{R^3}$$

(ii) If $x = R$

$$I = \frac{-GM}{R^2}$$

(iii) If $x > R$

$$I = \frac{-GM}{x^2}$$

* Relation b/w V_o an V_e

$$V_o = \sqrt{gR} \quad \text{and} \quad V_e = \sqrt{2gR}$$

so $\boxed{V_e = \sqrt{2}\, V_o}$